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TABLE OF CONTENTS

AUTHOR INDEX

(Continued on cover 4)

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HISTORY

*Yuktibhāṣā. Part I. Edited by A. R. Akhileśwara Iyer and Rāmavarma (Maru) Tampurān.** Mangalodayam Press, Trichur, 1948. viii+10+3+5+290+lxviii+19 pp. 10 Rupees. (Malayāḷam)

This is the first appearance in print of part of a Hindu mathematical work professing to be based on Tantrasaṅgraha, a Sanskrit work which is still in manuscript but widely accepted as the composition of a versatile scholar named Nīlakaṇṭha, whose period has been placed between 1450 A.D. and 1550 A.D. Yuktibhāṣā was first introduced to the English-speaking world by C. M. Whish [Trans. Roy. Asiatic Soc. 3, 509–523 (1835)]. Whish's work has been recently continued by K. Mukunda Marar and C. T. Rajagopal [J. Bombay Branch Roy. Asiatic Soc. N.S. 20, 65–82 (1944); these Rev. 6, 253; cited as I], A. Venkatraman [Math. Student 16, 1–7 (1949); these Rev. 11, 572; cited as II], and C. T. Rajagopal and A. Venkatraman [J. Roy. Asiatic Soc. Bengal. Sci. 15, 1–13 (1949); these Rev. 11, 572; cited as III].

The editors state in a foreword that they have used four manuscripts of Yuktibhāṣā. There are two other manuscripts available in the Madras Government Oriental Manuscipts Library: (i) MS in Malayāḷam bearing the title "Gaṇita-Yuktibhāṣā" and the number D332, (ii) MS in Sanskrit numbered R4382 and having the same title as (i). About these manuscripts the curator of the library has supplied the following information. MS (ii) ends with the explicit statement that it is incomplete; it is perhaps the original form of Yuktibhāṣā, though not all of it. MS (i) breaks off at the same point as (ii); and its first part, without the last two pages, would appear to be the book under review. The editors' foreword is followed by a preface by P. Śrīdhara Menon and a table of contents. The preface discusses the date and authorship of Yuktibhāṣā; but it is none too critical and not more informative than I, section III, and III, addendum.

Hindu mathematical works are generally written in verse and set forth rules and results without proofs. Yuktibhāṣā is an exception to the general practice, being a prose work which seeks to establish almost all its mathematical propositions. The subject matter of the printed portion of Yuktibhāṣā falls into two classes A, B. A: Some aspects of classical Hindu mathematics. B: Theorems of which the earliest mention is thought to be in Tantrasaṅgraha and Karaṇapaddhati. (The latter is an anonymous work supposed to have originated in the fifteenth century, now available in the Trivandrum Sanskrit Series.) The following are among the topics included in A. (a) Kuṭṭākāram, the Hindu solution of the linear indeterminate equation in two unknowns, which was first formulated by Āryabhaṭa I and later improved or altered or extended by others; (b) theorems involving chords of a circle, in particular, theorems on inscribed triangles and quadrilaterals. The last-mentioned theorems comprise some known to Brahmagupta, viz., (i) the expression for the circum-diameter of a triangle as the product of two sides of the triangle divided by the altitude on the third side, (ii) the result known as the Ptolemy's theorem for the cyclic quadrilateral, (iii) the formula $\{(s-a)(s-b)(s-c)(s-d)\}^{\frac{1}{2}}$ for the area of a cyclic quadrilateral of sides a, b, c, d and semi-perimeter s. Formula (iii) supplements another giving the area as the product of the three diagonals of the quadrilateral divided by twice its circum-diameter, where the diagonals are defined by a formula of Māhāvīrā. The propositions on chords of a circle are the geometrical equivalents of the trigonometric formulae $\sin(\theta\pm\phi)=\sin\theta\cos\phi\pm\cos\theta\sin\phi$, $\sin^2\theta-\sin^2\phi=\sin(\theta+\phi)\sin(\theta-\phi)$, of which the first can be found in Bhāskarā. The proofs of the results in A(b) with the exception of (iii), as given in Yuktibhāṣā, are reproduced in II. Perhaps two more results can be correctly classed among the above: (c) correct formulae for the surface and the volume of a sphere of given diameter. The theorems of class B are (a) the power series for $\tan^{-1}x$; (b) transformations of the series $\pi/4=1-1/3+1/5-\cdots$ into more rapidly convergent series and rational approximations to $\pi/4$ associated with the transformed series, all of them depending on a single idea which has been stated in general terms by C. T. Rajagopal [Scripta Math. 15, 201–209 (1949); these Rev. 11, 572]; (c) the power series for $\sin x$ and $\cos x$. A brief account of the topics in B(a) and B(b) appears in I. A more detailed account is given in two papers by C. T. Rajagopal and T. V. Vedamurti Aiyar, which are to appear in Scripta Math. An account of B(c) is presented in III.

Chapters I–IV of the printed part of Yuktibhāṣā are introductory in character. Chapter I treats of operations involving integers in eight sections on addition, subtraction, multiplication, division, squaring, square-root extraction, special combinations of the above operations:

$$(a\pm b)^2=a^2+b^2\pm 2ab,\quad (a+b)^2-(a-b)^2=4ab,$$
$$(a+b)^2+(a-b)^2=2a^2+2b^2,\quad a^2-b^2=(a+b)(a-b),$$

all of which are established geometrically as in Āryabhaṭa I, and their applications like

$$(a+b+c)^2=\sum a^2+2\sum bc,\quad 1+3+\cdots+(2n-1)=n^2.$$

Chapter II solves the ten pairs of simultaneous equations in x and y got by selecting any two of the five: $x+y=c_1$, $x-y=c_2$, $xy=c_3$, $x^2+y^2=c_4$, $x^2-y^2=c_5$, where the c's are constants. Chapter III extends to all rational numbers the operations restricted to integers in Chapter I. Chapter IV deals with the rule of three, direct proportion and inverse proportion.

In the remaining three chapters, V–VII, we have a mixture of number theory, analysis, and geometry. Chapter V is a connected account of kuṭṭākāram. It has two appendices (pages i–lxi at the end), one in Malayāḷam (based on Tantrasaṅgraha) and the other in English (introducing continued fractions). The problem of chapter V is niragrakuṭṭākāram which is in effect that of finding integral values

of x and y to satisfy the equation $bx-ay=c$, where a, b, c are known integers, $a>0$, $b>0$, $(a, b)=1$. This problem is solved by the method of kuṭṭaka (=cutter). The method consists in finding a part if not the whole of Euclid's algorithm for a/b, and exhibiting it as a valli (=creeper) whose elements run down two columns of a table. In page xliv of the English appendix the columns are marked I, II. Running up a third column, III, there is a second valli deduced from the first; and its two topmost elements are a pair of the values of x and y sought. A by-product of this process, as in Tantrasaṅgraha and Karaṇapaddhati, is the succession of convergents to the continued fraction for a/b. The convergents are not explicitly introduced as such; but their successive derivation, and their steady approach to a/b alternately from opposite sides, are recognized. Chapter V is to be judged in the light of modern inquiries into kuṭṭākāram [for instance, Saradakanta Ganguli, J. Indian Math. Soc., Notes and Questions **19**, 110–120, 129–142, 153–168 (1931–32)]. Chapter VI discusses the relation between the circumference and the diameter of a circle. It opens with the proof [reproduced in reference II] of the Pythagorean theorem for a right-angled triangle. The proof is essentially the same as a proof which, according to Bibhuti Bhusan Dutta ["Vedic mathematics," Cultural Heritage of India, Sri Ramakrishna Centenary Memorial, volume III], is traceable to the work Kātyāyana Śulba. The rest of the chapter is concerned with the circumference of a circle as the limit as $n\to\infty$ of the perimeter of the inscribed regular 2^n-gon, and the proofs of the results in B(a) and B(b). Chapter VII may be given the caption "Theorems on arcs and chords of a circle," if we leave out the formulae at its end for the surface and the volume of a sphere. The theorems are, in the order of their appearance, those of B(c), the theorem

$$\sin^2 x = x^2 - \frac{x^4}{(2^2-2/2)} + \frac{x^6}{(2^2-2/2)(3^2-3/2)} - \frac{x^8}{(2^2-2/2)(3^2-3/2)(4^2-4/2)} + \cdots,$$

and finally the theorems of A(b).

The division of the work under review into chapters and articles, each with a separate title, the diagrams, the footnotes and the explanations inserted within brackets, are all due to the editors, and are essential for easy reading. The editors have also provided at the end (i) a list in Malayāḷam and English of the propositions proved, which is incomplete since it does not include the propositions in B, (ii) a key to the mathematical terminology used, in two glossaries, Malayāḷam-English and English-Malayāḷam. A welcome addition to the two appendices on kuṭṭākāram would have been a third examining the material common to Yuktibhāṣā and Kriyākramakarī, a Sanskrit manuscript of unknown authorship apparently inspired by Tantrasaṅgraha and contemporaneous with it [cf. the information supplied by one of the present editors of Yuktibhāṣā in reference I, appendix III]. *C. T. Rajagopal* (Tambaram) and *A. Venkatraman* (Kozhikode).

★Bhāskara. Laghubhāskarīyam. With Parameśvara's commentary. Ānandāśram Sanskrit Series, no. 128. Ānandāśram Press, Poona, 1946. iv+16+92+5+3 pp.

This astronomical (and mathematical) work is by Bhāskara I, who lived about A.D. 512, and who was a pupil of Āryabhaṭa and the first commentator on the Āryabhaṭīya. The author is not the Bhāskara who wrote the Siddhāntaśiromaṇi, for he lived about A.D. 1150. An introduction to the book gives particulars concerning the several Hindu astronomers and their works. The text itself consists of 216 verses grouped in 8 chapters. With the exception of the last chapter on asterisms, the titles of the chapters are the same as chapters I, II, III, IV, V, X, and IX of the translation by Burgess of the Sūryasiddhānta. The text is accompanied by a commentary by Parameśvara who lived A.D. 1408 and who is, presumably, the same person as commentated upon the Āryabhaṭīya by Āryabhaṭa. This work is not listed by Thibaut [Astronomie, Astrologie und Mathematik, Grübner, Strassburg, 1899] nor by Winternitz [Geschichte der indischen Litteratur, Leipzig, 1920, volume 3]. The only references known to the reviewer are in Datta and Singh [History of Hindu Mathematics, Lahore, v. I, 1935, v. II, 1938] and in the paper by Balagangadharan [Math. Student **15**, 55–70 (1947); these Rev. **10**, 667]. *E. B. Allen.*

★Bhāskara. Mahābhāskarīyam. With Parameśvara's commentary called Karmadīpikā. Ānandāśram Sanskrit Series, no. 126. Ānandāśram Press, Poona, 1945. iii+8+92+8+4 pp.

This is another astronomical (and mathematical) work by Bhāskara I [see the preceding review], and consists of 395 verses occurring in 8 chapters. The first three of these chapters have the same titles as the first three of the Sūryasiddhānta and one chapter deals with eclipses. The other chapters bear the titles sphuṭāvidhiḥ, darśanasaṁskārādihiḥ, yugabhagaṇādhiḥ, and tithyānayanādhiḥ. The commentary of Parameśvara on the text is included in the book and data on Hindu astronomers and their works occur in the introdution. This work is not mentioned by Thibaut [see the preceding review for this and the following references] nor Winternitz but is listed by Datta and Singh and by Balagangadharan. *E. B. Allen* (Troy, N. Y.).

Frajese, Attilio. Sul valore di un'attribuzione a Platone della conoscenza di due poliedri semiregolari. Archimede **2**, 89–95 (1950).

Hero in his Definitions [Opera, v. IV, p. 66, Teubner, Leipzig, 1912] attributes to Plato the knowledge of two semiregular polyhedra, one of which, however, does not exist. The author contends that Plato alludes to the other one (which is contained by six squares and eight equilateral triangles) in Timaeus (56d) when discussing the mixture of the particles of the element earth with those of one of the other elements. *E. J. Dijksterhuis* (Oisterwijk).

Frajese, A. Su alcune questioni della storia della matematica greca. Archimede **1**, 41–47 (1949).

The author summarizes the most important problems of the history of Greek mathematics which still await their final solution: the origin of Proclus's Summary; the discovery of the irrational; the correct date of the Pythagorean mathematical achievements; the relations between Greek mathematics and Greek philosophy. *E. J. Dijksterhuis.*

Frajese, A. Sul significato dei postulati euclidei. Scientia **44**, 299–305 (1950).

Itard, Jean. Quelques remarques sur les méthodes infinitésimales chez Euclide et Archimède. Rev. Hist. Sci. Appl. **3**, 210–213 (1950).

Archibald, R. C. The first translation of Euclid's elements into English and its source. Amer. Math. Monthly **57**, 443–452 (1950).

***Euclid's Elements, Book II. Translated and annotated by Anton Bilimović.** Srpska Akademija Nauka. Klasični Naučni Spisi, Kn. II. Matematički Institut, Kn. 2. Belgrade, 1950. 29 pp. (Serbo-Croatian)

Hjelmslev, Johannes. Eudoxus' axiom and Archimedes' lemma. Centaurus **1**, 2–11 (1950).

Drachmann, A. G. Heron and Ptolemaios. Centaurus **1**, 117–131 (1950).

Gandz, Solomon. The astronomy of Maimonides and its sources. Arch. Internat. Hist. Sci. (N.S.) **3**, 835–855 (1950).

Greenwood, Thomas. L'esprit aristotélicien dans les mathématiques modernes. Rev. Trimest. Canad. **36**, 380–386 (1950).

Procissi, Angiolo. Nuove ricerche bibliografiche galileiane. Boll. Un. Mat. Ital. (3) **5**, 170–174 (1950).

Moody, Ernest A. Laws of motion in medieval physics. Scientific Monthly **72**, 18–23 (1951).

Hondl, Stanko. Stay and Bošković on absolute motion. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **5**, 21–32 (1950). (Croatian. English summary)

Pihl, Mogens. A small contribution to the history of the lever law. Mat. Tidsskr. B. **1950**, 123–127 (1950). (Danish)

Discussion of a proof of the lever law by Roemer as revealed by a manuscript of Leibniz. *O. Neugebauer.*

Vavilov, S. I. I. Newton's "Lectures on Optics." Akad. Nauk SSSR. Trudy Inst. Istorii Estestvoznaniya **1**, 315–326 (1947). (Russian)

Loria, Gino. La storia della matematica vista da un veterano. Boll. Un. Mat. Ital. (3) **5**, 165–170 (1950).

Slessenger, W. W. O., and Curtis, A. R. A very short history of mathematics. Math. Gaz. **34**, 82–83 (1950).

Turán, Pál. 30 years of mathematics in the Soviet Union. III. Results of number-theory in the Soviet Union. Mat. Lapok **1**, 243–266 (1950). (Hungarian. Russian and English summaries)

Kadeřávek, Fr. Une notice historique sur la géométrie descriptive dans nos pays tchèques. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. **1949**, no. 15, 7 pp. (1950). (Czech. French summary)

Harkin, Duncan. The scientific contributions of Niels-Henrik Abel (1802–1829) and Sophus Lie (1842–1899). Norsk Mat. Tidsskr. **32**, 68–78 (1950).

Erdélyi, A. Obituary: Harry Bateman (1882–1946). Obit. Notices Roy. Soc. London **5**, 591–618 (1 plate) (1947).

Thorndike, Lynn. Giovanni Bianchini in Paris manuscripts. Scripta Math. **16**, 5–12, 169–180 (1950).

Bernard-Maitre, Henri. Un historien des Mathématiques en Europe et en Chine: Le Père Henri Bosmans, S.J. (1852–1928). Arch. Internat. Hist. Sci. (N.S.) **3**, 619–628 (1950).

Bernard-Maitre, Henri. Bibliographie des œuvres du Père H. Bosmans. Arch. Internat. Hist. Sci. (N.S.) **3**, 629–656 (1950).

Viguier, Gabriel. Adolphe Buhl. Bull. Soc. Math. Phys. Macédoine **1**, 83–87 (1950).

Tietze, Heinrich. Obituary: Constantin Carathéodory. Arch. Math. **2**, 241–245 (1950).

List of mathematical papers of L. Fejér. Mat. Lapok **1**, 267–272 (1950).

Speiser, A. Obituary: Rudolf Fueter. Elemente der Math. **5**, 98–99 (1950).

Fueter, Rudolf. Nachgelassene Aufzeichnungen. Elemente der Math. **5**, 99–104 (1950).

A brief autobiography of the author up to April, 1946.

Yuškevič, A. P. Academician S. E. Gur'ev and his rôle in the development of Russian science. Akad. Nauk SSSR. Trudy Inst. Istorii Estestvoznaniya **1**, 219–268 (1947). (Russian)

Titchmarsh, E. C. Obituary: Godfrey Harold Hardy (1877–1947). Obit. Notices Roy. Soc. London **6**, 447–461 (1 plate) (1949).

Watson-Watt, Robert. Oliver Heaviside: 1850–1925. Scientific Monthly **71**, 353–358 (1950).

Fog, David. Obituary: Johannes Hjelmslev (1873–1950). Mat. Tidsskr. A. **1950**, 1–20 (1950). (Danish)

Milne, E. A. Obituary: James Hopwood Jeans (1877–1946). Obit. Notices Roy. Soc. London **5**, 573–589 (1 plate) (1947).

Polubarinova-Kočina, P. Ya. The scientific work of S. V. Kovalevskaya. Uspehi Matem. Nauk (N.S.) **5**, no. 4(38), 3–14 (1 plate) (1950). (Russian)

Grioli, Giuseppe. L'opera scientifica di Ernesto Laura. Rend. Sem. Mat. Univ. Padova **19**, 443–449 (1 plate) (1950).

Obituary: N. N. Luzin. Uspehi Matem. Nauk (N.S.) **5**, no. 4(38), 15–18 (1 plate) (1950). (Russian)

Agostini, Amedeo. L'opera matematica di Pietro Mengoli. Arch. Internat. Hist. Sci. (N.S.) **3**, 816–834 (1950).

Agostini, Amedeo. L'opera matematica di Pietro Mengoli. Archimede **2**, 165–170 (1950).

Godeau, Robert. Obituary: Adolphe Mineur. Bull. Soc. Math. Belgique **2** (1948–1949), 10–14 (1950).

Matsumoto, Toshizô. Obituary: Hiroshi Okamura. Mem. Coll. Sci. Univ. Kyoto Ser. A. **26**, 1–3 (1950).

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List of mathematical papers of F. Riesz. Mat. Lapok **1**, 273–277 (1950).

Yanovskaya, S. A. Michel Rolle as a critic of the infinitesimal analysis. Akad. Nauk SSSR. Trudy Inst. Istorii Estestvoznaniya **1**, 327–346 (1947). (Russian)

Aleksandrov, P. S. Obituary: Vyačeslav Vasil'evič Stepanov. Uspehi Matem. Nauk (N.S.) **5**, no. 5(39), 3–10 (1 plate) (1950). (Russian)

Obituary: Vyačeslav Vasil'evič Stepanov. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 565–568 (1 plate) (1950). (Russian)

Mambriani, A. Obituary: Leonida Tonelli. Rivista Mat. Univ. Parma **1**, 157–188 (1 plate) (1950).

Kasner, Edward, and Harrison, Irene. Voltaire on mathematics and horn angles. Scripta Math. **16**, 13–21 (1950).

Taylor, H. S. Obituary: Joseph Henry Maclagen Wedderburn (1882–1948). Obit. Notices Roy. Soc. London **6**, 619–625 (1 plate) (1949).

Archibald, R. C. New information concerning Isaac Wolfram's life and calculations. Math. Tables and Other Aids to Computation **4**, 185–200 (1950).

ALGEBRA

Touchard, Jacques. Contribution à l'étude du problème des timbres poste. Canadian J. Math. **2**, 385–398 (1950).

Ce problème a été proposé par Lemoine [E. Lucas, Théorie des nombres, tome 1, Gauthier-Villars, Paris, 1891, p. 120]: Quel est le nombre, P_n, des manières de replier sur un seul une bande de n timbres-poste? Il a été étudié, entre autre, par J. J. van Laar [Arch. Musée Teyler (2) **8**, 13–71 (1902)] et par le rapporteur [Sur les chevauchements des permutations, Marseille, 1949; ces Rev. **11**, 153] sans qu'aucune solution générale ait pu être obtenue. (I) Si, dans le schéma de Ste.-Laguë [Les réseaux, Mémor. Sci. Math., no. 18, Gauthier-Villars, Paris, 1926, p. 39] on rabat une demi-courbe sur l'autre, les points d'intersection qui apparaissent sont appelés "points doubles fictifs" (pdf). Expression de T_0, T_1, T_2, où T_k est le nombre des schémas de n éléments avec k pdf. (II), (III) Généralisation des résultats du rapporteur. La formule: $A(n, 2k) = A(n, 2k-1)$, $n \geqq 2k$, où $A(n, t)$ est le nombre des solutions commençant par 1 et où t occupe la 2^e place, est démontrée pour la première fois: correspondance une-à-une entre les deux formes et construction par laquelle on passe de la 1^e à la 2^e. Soit P une solution dans laquelle $E = a, a+1, \cdots, b$, est un ensemble d'éléments qui se suivent dans un ordre quelconque, mais sans intercalation d'éléments étrangers à E. Soit $R(a, b)$ l'opération consistant à renverser l'ordre de E sans toucher aux termes qui suivent ou précèdent E; par ex., $P = 128976453$, $P \cdot R(6, 9) = 126798453$. Les opérations $R(a, n)$, $a = 1, \cdots, n$, qui ne sont pas des substitutions, engendrent un groupe isomorphe à $(G_2)^{n-2}$. (IV) Recherches sur le problème des demi-schémas [C. R. Acad. Sci. Paris **230**, 1997–1998 (1950); ces Rev. **12**, 44]. (V) Tables de P_n, T_k, et des résultats du (IV). *A. Sade* (Marseilles).

Saxena, P. N. A simplified method of enumerating Latin squares by MacMahon's differential operators. I. The 6×6 Latin squares. J. Indian Soc. Agric. Statistics **2**, 161–188 (1950).

The author simplifies MacMahon's method of differential operators sufficiently to make an enumeration of the 6×6 reduced Latin-squares possible. His result (9408) checks with Tarry, who obtained this number first by tactical enumeration [Assoc. Franç. Avancement Sci. C. R. **1900**$_2$, 170–203]. The author hopes to derive the number of reduced 7×7 squares in a subsequent publication.
H. B. Mann (Columbus, Ohio).

Šilov, G. E. An attempt to present the theory of determinants without the theory of substitutions. Uspehi Matem. Nauk (N.S.) **5**, no. 5(39), 177–179 (1950). (Russian)

Mitrinovitch, D. S. Remarques sur des déterminants du type d'Escherich. Bull. Soc. Math. Phys. Macédoine **1**, 1–20 (1950). (Serbo-Croatian. French summary)

A determinant is said to be of Escherich's type if $a_{0k} = a_k$, $a_{kk} = x_k$, $a_{k+1,k} = -y_{k+1}$, while all other $a_{jk} = 0$, $k = 0, \cdots, n$, $x_0 = a_0$ [cf. E. Pascal, I determinanti, 2d ed., Hoepli, Milan, 1923, p. 215]. The author puts in particular $a_k = x^k$, $x_0 = 1$, $x_k = k$ for $k \geqq 1$ and $y_k = \lambda - (n-1)$. He shows that the determinant equals $n! \sum \binom{\lambda}{k} x^k$ and generalizes this result.
W. Feller (Princeton, N. J.).

Hirsch, K. A. On the generalised Vandermonde determinant. Math. Gaz. **34**, 118–120 (1950).

Sanielevici, S. Rotations in spaces of n dimensions. Acad. Repub. Pop. Române. Bul. Şti. A. **1**, 661–669 (1949). (Romanian. Russian and French summaries)

Algebraically, the author's result on rotations in n-dimensional space is merely to the effect that in order for a skew symmetric matrix to have $n-2$ zero characteristic roots it must be of rank 2. *D. C. Lewis* (Baltimore, Md.).

Ledermann, Walter. Bounds for the greatest latent roots of a positive matrix. J. London Math. Soc. **25**, 265–268 (1950).

Let $a_{ij} > 0$, $i, j = 1, \cdots, n$, and ω be the root of maximum modulus of $A = \|a_{ij}\|$; ω is then real, positive, simple and has an associated eigenvector with all components positive. Moreover, $\omega \leqq R = \max R_i$, where $R_i = \sum_j a_{ij}$, with equality holding if and only if the row sums R_i are all equal. This result is improved in case not all the R_i are equal: $\omega < R - \kappa(1 - \delta^{\frac{1}{2}})$, where $a_{ij} \geqq \kappa > 0$ and $1 > \delta =$ maximum of $R_i(R_j)^{-1}$ for $R_i < R_j$. Also, $\omega > r + \kappa(\delta^{-\frac{1}{2}} - 1)$, where $r = \min R_i$.
W. Givens (Knoxville, Tenn.).

Sce, Michele. Osservazioni sulle forme quasi-canonica e pseudo-canonica delle matrici. Rend. Sem. Mat. Univ. Padova **19**, 324–339 (1950).

Cherubino [Ann. Scuola Norm. Super. Pisa (3) **2** (1948), 151–166 (1950); these Rev. **11**, 489] studied the quasi-canonical form of a matrix obtained by unitary transformation. There the matrix is transformed to triangular form with the characteristic roots along the principal diagonal in a special order in which equal roots are not separated. If this last condition is omitted, the form is called pseudo-canonical. Sufficient conditions are obtained in order that the same unitary matrix can transform $p-1$ of p matrices A_i into pseudo-canonical and one of them into quasi-canonical form. The condition concerns the matrices $A_iA_k - A_kA_i$ and

includes permutability of the A_i as a special case. The following generalization of a theorem of Frobenius concerning commutative matrices is obtained: Let $f(x_1, \cdots, x_p)$ be a rational function and $A_1, \cdots, A_p$ matrices which can be transformed into pseudo-canonical form by the same unitary matrix. The characteristic roots of $f(A_1, \cdots, A_p)$ are then of the form $f(x_1, \cdots, x_p)$, where x_i is a suitably chosen characteristic root of A_i. *O. Todd-Taussky.*

Bosshard, Paul. **Die Cliffordschen Zahlen, ihre Algebra und ihre Funktionentheorie.** Thesis, University of Zürich, 1940. 48 pp.

This paper considers the algebraic properties of Clifford algebras over the field of all real numbers and the related function theory for a special case. In the first chapter the algebraic properties of Clifford algebras are presented with particular emphasis on the case $n=4$, where the algebra is the set $M_2(Q)$ of all two-rowed matrices with real quaternionic elements. A Clifford variable is then a variable in $M_2(Q)$ and the notion of a continuous function of a Clifford variable is defined. Differentials and integrals are considered. The concept of a regular Clifford function is introduced and the Cauchy-Riemann equations for regularity are derived. Two Cauchy integral theorems are derived and a related study is made of the Dirac differential equations of electron theory. *A. A. Albert* (Chicago, Ill.).

Abstract Algebra

Petropavlovskaya, R. V. On laws of structures. Doklady Akad. Nauk SSSR (N.S.) 74, 661–662 (1950). (Russian)

Let $T_{n+1}=\{[\{(x\cup y)\cap T_n\}\cup y]\cap x\}\cup T_n$ and $T_0=z$. Then in any lattice $T_n=T_{n-1}$ implies $T_{n+1}=T_n$; given n, there is a lattice in which $T_{n+1}=T_n$ for all elements x, y, z, but $T_n\neq T_{n-1}$ for some x, y, z. [The author's theorem 2 is already known; cf. G. Birkhoff, Lattice Theory, Amer. Math. Soc. Colloquium Publ., vol. 25, revised ed., New York, 1948, p. 20, example 7a; these Rev. **10**, 673.] *P. M. Whitman* (Silver Spring, Md.).

Trevisan, Giorgio. A proposito delle relazioni di congruenza sui quasi-gruppi. Rend. Sem. Mat. Univ. Padova **19**, 367–370 (1950).

It is proved that the normal congruence relations on a quasigroup are permutable, and hence, since a finite quasigroup has only normal congruence relations, that all congruence relations on a finite quasigroup are permutable. This last is a partial solution of a problem stated by Birkhoff [Lattice Theory, Amer. Math. Soc. Colloquium Publ., vol. 25, 2d ed., New York, 1948, p. 86, problem 31; these Rev. **10**, 673]. The paper is self-contained, and the methods are elementary. *F. Kiokemeister* (South Hadley, Mass.).

Matsushita, Shin-ichi. L'algèbre des opérations topologiques. II. J. Osaka Inst. Sci. Tech. Part I. 1, 77–80 (1949).

As in part I of this paper [Math. Japonicae 1, 28–35 (1948); these Rev. **10**, 348] the author considers a Boolean algebra with abstract derivation operator X^d and closure operator $X^c=X+X^d$. He now defines $X^\gamma=X^{dcdcd}$, $\bar{X}=X\cdot 1^{dcdc}$, and $X^\beta=\bar{X}+\bar{X}^\gamma$, obtains a number of identities, and shows that each of the following is a necessary and sufficient condition that X be nondense: (α) $X^{\beta c\beta c\beta}=\bar{0}$; ($\beta$) $X^{\beta c\beta c\beta}=E^d$, where $E=1^{dc}$. *I. Halperin.*

Shôda, Kenjirô. **Daisûgaku-tsûron. [General Algebra].** Kyôritsu-shuppan, Tokyo, 1947. 2+2+226+5 pp. 110 Yen.

In this book a systematic and consistent treatment of general algebraic systems is given, which includes in particular the results of the author's previous work and a part of his recent paper [Osaka Math. J. 1, 182–225 (1949); these Rev. **11**, 308 and references cited there]. In the first chapter the fundamental notions are introduced and discussed. An algebraic system is defined as a set possessing a family of compositions (where a composition may not have meaning for all pairs of elements), a primitive algebraic system as one in which every composition is defined for every pair of elements and which admits certain identities with respect to compositions, while an elementary algebraic system is a weakening of the latter in which the identities are supposed to be valid as soon as both sides have meaning. The incorporation of mappings and compositions of many elements into the author's theory is discussed. Employing a rather strong kind of homomorphism notion (in which a pair can be composed when the corresponding pair can be composed) the theorems of homomorphism and of meromorphism are established. A subsystem is called normal when it is a class in a congruence relation. Assuming the existence of an element, called null, which is idempotent for every composition, the 1st and the 2d isomorphism theorems are established; particular caution is required since a normal subsystem does not determine a congruence, or residue-system, uniquely. Lattices, groups, groupoids, mixed groups (of Loewy) are considered. For instance, the notion of group is shown to be primitive by taking division as its composition.

The second chapter is on the theory of free systems, including the fundamental theorem and the theorem of change of generators (of Tietze). A theory of independence is given, making use of a certain notion of valuation so as to take care of algebraic and linear dependence, the latter being distinguished in that an element is (linearly) dependent on a set of elements if and only if it is contained in the subsystem generated by the set. Free lattices, free groups, free Lie and associative rings are treated, emphasizing the relationship among them.

The third chapter starts by proving the author's sufficient condition for a lattice of (all) congruences to be modular (that every meromorphism between two systems homomorphic to our system is a "class-to-class" one); a different form of this fact is given by Birkhoff [Lattice Theory, Amer. Math. Soc. Colloquium Publ., v. 25, 2d ed., New York, 1948; these Rev. **10**, 673]. Then the author assumes for his algebraic systems the existence of null elements and that the join of two normal subsystems is always normal, besides the above condition of "class-to-class" meromorphism, and he develops generalized theories of normal chains, composition series, of direct and subdirect products, and generalizations of the Jordan-Hölder and the Remak-Schmidt-Ore theorems. After completely reducible systems, the notions of solvable and nilpotent systems are discussed, where general identities are considered instead of the usual commutativity. Further, a set of endomorphisms of an algebraic system is considered as an algebraic system in terms of the usual multiplication (of mappings) and the compositions induced from those of the original system. Here the multiplication is of course associative, and distributive for the latter compositions, presenting the notion of ring-systems as a generalization of the ring notion. Structural theory of abstract ring-systems is developed,

under chain conditions, including (generalized) Peirce decompositions and Wedderburn's theorem; for the latter the notion of matrices is also generalized.

The last chapter gives the theory of representing (primitive) algebraic systems as systems of endomorphisms of some other systems called representation systems. Reduction and direct decomposition are discussed in connection with representation systems. Particular observations are made to the cases of ring-systems, rings, and groups.

In spite of the very general treatment, the author tries, with success, to give theorems which are by themselves significant for individual and concrete cases.

T. Nakayama (Urbana, Ill.).

Šneĭdmyuller, V. I. Infinite rings with finite decreasing chains of subrings. Mat. Sbornik N.S. **27**(69), 219–228 (1950). (Russian)

The author improves his earlier results on rings with the descending chain condition on subrings [C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 579–581 (1940); these Rev. **2**, 121]. He first proves three basic facts about such a ring R: (1) R is locally finite (every finitely generated subring is finite); (2) if R is a division ring, it is an absolutely algebraic field of characteristic p; (3) if R is completely primary but not a field, then R is finite. This leads to a theorem asserting that R is the direct sum of a finite number of rings, each of which is a field, or a ring extension of a trivial ring by a finite ring. In the final section a connection is made with the theory of nilpotent groups. *I. Kaplansky* (Chicago, Ill.).

Snapper, E. Completely primary rings. I. Ann. of Math. (2) **52**, 666–693 (1950).

This is the first of a series of four papers in which the author proposes to extend a large part of the Steinitz field theory to completely primary rings. The ideas have their origin in fundamental work of Krull [Math. Ann. **88**, 80–122 (1922); **91**, 1–46 (1924); **92**, 183–213 (1924)]. The rings considered are all commutative rings with unit element. If A is a ring, let N denote the radical of A, that is, N consists of the nilpotent elements of A. A considerable part of this paper consists, of a careful exposition, in as general a form as possible of the underlying concepts; but a number of results are obtained that are of interest in themselves. Consistent use is made of the natural homomorphism of A onto A/N and, in particular, of the relations which hold regarding factorizations in A and A/N. An element α of A is a fundamental irreducible of A if its image $\bar{\alpha}$ in A/N is an irreducible element of A/N. A nontrivial irreducible is an irreducible element which is not a unit. If f is a regular element of the polynomial ring $A[x]$, the order of f is defined to be the minimum degree of the nonzero polynomials of the principal ideal (f). It is proved that if f is a regular polynomial, there exists a nonzero element α of A such that the order of f is equal to the degree of αf. If A has no nonzero nilpotent elements and f is regular, then no regular polynomial of (f) has degree less than the degree of f. A polynomial f is a unit of $A[x]$ if and only if its constant term is a unit of A and all other coefficients are nilpotent. The Jacobson radical of $A[x]$ consists simply of the nilpotent elements of $A[x]$. The principal ideal (f) can be generated by a monic polynomial of degree m if and only if in f the coefficient of x^m is a unit of A and the coefficients of all higher powers of x are nilpotent. Now let R be a completely primary ring, and f a regular element of $R[x]$. Using results of which the ones stated above are typical, it is shown that $f=e(x)\prod[e_i(x)(p_i(x))^{k_i}+n_i(x)]$, where $e(x)$ and the $e_i(x)$ are units of $R[x]$, the $p_i(x)$ are nontrivial fundamental irreducibles of $R[x]$, and the $n_i(x)$ are nilpotent elements of $R[x]$. Actually, the above factorization is a factorization of f into primary elements (an element α is primary if the principal ideal (α) is primary), and is unique in the ideal theory sense. Various other miscellaneous results are obtained in preparation for later papers of the series. *N. H. McCoy* (Northampton, Mass.).

Golovina, L. I. Commutative radical rings. Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 449–472 (1950). (Russian)

Let R be a commutative ring, Q a subring. The element x in R is integral over Q if it satisfies an equation $x^k+b_1x^{k-1}+\cdots+b_{k-1}x+c=0$, where $c\epsilon Q$ and b_i is an integer plus an element of Q. If the b's are actually in Q, then x is completely integral over Q. The author proves the transitivity of these relations, and shows they are inherited by sums and products. Next, adjunctions to radical rings are studied (R is a radical ring if it is a group under the operation $x+y-xy$). The following two theorems are typical. If Q is a radical ring and x is completely integral over Q, then $Q(x)$ is a radical ring. If Q is a nil ring and $Q(x)$ a radical ring, then x is nilpotent. In the last part of the paper, R is a commutative radical ring without divisors of 0. It is proved that R possesses a unique maximal radical extension R^* with R^* completely integral over R; R^* consists of all completely integral elements of the algebraic closure of the quotient field of F. Two other kinds of radical algebraic closure are studied. As a corollary the author shows that a field is a quotient field of a radical ring if and only if it is not absolutely algebraic of characteristic p. All radical subrings of the rational numbers are determined; they are of the form $R_k=$ set of all $km/(kn+1)$, where k is a fixed integer with $|k|>1$, and m and n run over all integers. There is some overlapping with recent work of Snapper [see the preceding review]. *I. Kaplansky* (Chicago, Ill.).

Žukov, A. I. Reduced systems of defining relations in nonassociative algebras. Mat. Sbornik N.S. **27**(69), 267–280 (1950). (Russian)

Let F be a free nonassociative algebra and W a two-sided ideal in F. A basis w_i of W is said to be reduced if one can pick a monomial s_i in each w_i in such a way that no two s_i's divide each other, and no other term of a w_i is divisible by any s_j. It is shown by a transfinite induction that any ideal has such a reduced basis. Then this is used to prove the following results concerning an algebra $G=F/W$ given by generators and relations. (1) A criterion is given for determining when two elements of G are equal. (2) If G is infinite-dimensional and has a finite number of generators and relations, then G contains a free subalgebra. (3) If G is generated by $g_1, \cdots, g_n$, subject to one relation actually involving g_1, then the subalgebra generated by $g_2, \cdots, g_n$ is free (this is the analogue of the "Freiheitssatz" of group theory). (4) If G has countable dimension, it can be embedded in an algebra with one generator.

I. Kaplansky (Chicago, Ill.).

Azumaya, Gorô. Corrections and supplementaries to my paper concerning Krull-Remak-Schmidt's theorem. Nagoya Math. J. **1**, 117–124 (1950).

The author makes some corrections on the portion of his paper [Jap. J. Math. **19**, 525–547 (1948); these Rev. **11**, 316] dealing with an infinite Krull-Schmidt theorem. In

the corrected version he is able to drop, in part, the hypothesis that the indecomposable summands are finitely generated. *I. Kaplansky* (Chicago, Ill.).

★Büsser, Albert Heinrich. Über die Primidealzerlegung in Relativkörpern mit der Relativgruppe $\mathfrak{G}_{168}$. Thesis, University of Zürich, 1944. 41 pp.

The author carries through in all detail the investigation of a field K, whose Galois group with respect to an algebraic number field k is the simple group of order 168, and also discusses certain intermediate subfields with Galois groups of orders 24, 21, and 8. There are twenty-two different possibilities for the manner in which a prime ideal $\mathfrak{p}$ of k may decompose in K and its subfields, distinguished by different values of the integers E and F, where E and EF are respectively the orders of the inertia group and the decomposition group of a prime ideal of K that divides $\mathfrak{p}$. Having found invariants f, Δ, C, K of degrees 4, 6, 14, and 21 in the variables of a ternary representation of $\mathfrak{G}_{168}$, and having introduced fractional invariants $\kappa=-C^3/\Delta^7$ and $\mu=f\Delta^4/C^2$, the author determines explicit resolvent equations belonging to the various subgroups of $\mathfrak{G}$. In a final section of the paper a study is made of the way in which prime ideals relatively prime to 168 decompose in K, depending on number-theoretic properties of the factors κ^{-1} and $\kappa-12^3$ of the discriminant. These examples illustrate half of the twenty-two cases. The other cases involve the primes 2, 3, 7. *J. S. Frame* (Princeton, N. J.).

Hochschild, G. Automorphisms of simple algebras. Trans. Amer. Math. Soc. **69**, 292–301 (1950).

L'auteur développe une théorie de Galois des algèbres simples de rang fini sur leur centre, en reprenant la question à l'origine, et sans établir aucune connection entre ses résultats et ceux obtenus antérieurement par le rapporteur [Comment. Math. Helv. **21**, 154–184 (1948); ces Rev. **9**, 563; nous désignerons ce mémoire par TG]. Dans la terminologie de TG, on peut présenter les résultats de l'auteur comme suit. Soit E un espace vectoriel à gauche de dimension finie sur un corps K (commutatif ou non) de rang fini sur son centre Z; soit A l'algèbre simple des endomorphismes de E, A et K étant considérés comme plongés dans l'anneau $\mathcal{E}$ des endomorphismes du groupe abélien E; on sait que tout automorphisme de A est de la forme $v\rightarrow uvu^{-1}$, où u est un semi-automorphisme de l'espace vectoriel E, relatif à un automorphisme du corps K. Les résultats nouveaux (par rapport à TG) obtenus par l'auteur sont en substance les suivants: (I) Soit R un sous-anneau simple de A contenant Z; soient u_λ des semi-automorphismes de E en nombre fini, comprenant l'identité, tels que $u_\lambda Ru_\lambda^{-1}=R$, et que $u_\lambda u_\mu^{-1}=xu_\nu$ pour un indice ν convenable et pour un $x\epsilon KR$, avec la condition $u_\lambda\bar{\epsilon}KA$ si $\lambda\neq\mu$. Dans ces conditions, l'anneau C engendré (dans $\mathcal{E}$) par K, R, et les u_λ est simple (cela découle du raisonnement du lemme 1.2, pp. 293–294, joint au lemme 2 de TG; le théorème 2.1 en résulte aussitôt, compte tenu de TG). (II) Soit B un sous-anneau simple de A, tel que le degré de A sur B soit fini et que BZ soit simple; soit C le sous-anneau de $\mathcal{E}$ engendré par tous les semi-automorphismes de E qui permutent avec les éléments de B; alors C est engendré comme dans (I) (il n'en résulte peut-être pas nécessairement que le bicommutant de B dans $\mathcal{E}$ soit galoisien dans A, au sens de TG). (III) Soit L un sous-anneau simple fortement galoisien de A, tel que le degré de A sur L soit fini, et soit B un sous-anneau simple de A contenant L et tel que BZ soit simple; alors B est fortement galoisien dans A. Un exemple de Teichmüller montre que ce dernier résultat ne subsiste plus lorsque BZ n'est pas supposé simple, ce qui (compte tenu du théorème 4 de TG) fournit un exemple de sous-anneau galoisien mais non fortement galoisien. Enfin, l'auteur donne des conditions moyennant lesquelles, dans les hypothèses de (III), L est fortement galoisien par rapport à B.

J. Dieudonné (Baltimore, Md.).

Hochschild, G. Note on Artin's reciprocity law. Ann. of Math. (2) **52**, 694–701 (1950).

Let K be finite algebraic over the rational field, J_K the group of K-idèles [Chevalley, same Ann. (2) **41**, 394–418 (1940); these Rev. **2**, 38] and K^* the multiplicative group of K, considered as subgroup of J_K. Let L be an Abelian extension of K. Using local class field theory in the form recently developed by the author [ibid. **51**, 331–347 (1950); these Rev. **11**, 490] it is easy to define a homomorphism of J_K into the Galois group of L/K, such that $N_{L/K}J_L$ is contained in the kernel. Artin's reciprocity law is the statement that the kernel is exactly $K^*\cdot N_{L/K}J_L$ and that every element of the Galois group is an image. A proof of this law is given, assuming the index theorems [as proved by Chevalley, loc. cit.]: norm index (i.e., $[J_K:K^*\cdot N_{L/K}J_L]$) of an Abelian extension divides the degree; for cyclic extensions, norm index equals degree, and every element of K which is everywhere local norm is a norm. This proof follows the same plan (reduction to a cyclotomic case) as Hasse's [Math. Ann. **107**, 731–760 (1933); also in Deuring, Algebren, Springer, Berlin, 1935, VII, § 5.6]. But it is much simplified by disentangling the reciprocity law from other theorems on algebras. Notably, the Sylow group argument is avoided since only cocycles corresponding to cyclic algebras are needed.

G. Whaples (Bloomington, Ind.).

Dem'yanov, V. B. On cubic forms in discretely normed fields. Doklady Akad. Nauk SSSR (N.S.) **74**, 889–891 (1950). (Russian)

Hasse has shown [J. Reine Angew. Math. **152**, 129–148 (1923)] that every quadratic form in more than 4 variables over the field Q_p of p-adic numbers has a nontrivial zero in Q_p. Mordell has shown [J. London Math. Soc. **12**, 127–129 (1937)] that for any $n>0$ there exists a form over Q_p of degree n in n^2 variables which has no nontrivial zero in Q_p. There arose thus the conjecture that every form over Q_p of degree n in n^2+1 variables has a nontrivial zero in Q_p. In the present note the author verifies this conjecture for $n=3$, $p\neq 3$. The first step is to prove the lemma that if a cubic form F over a field K of characteristic $\neq 3$ has no nontrivial zero in K, then F can be transformed by a reversible linear transformation over K into a form in which every $x_i^2x_j$ $(i<j)$ has coefficient 0. Next the author considers any field K complete with respect to a discrete valuation and with residue class field P of characteristic $\neq 3$. Using the lemma he shows that if every cubic form over P in more than t variables has a nontrivial zero in P, then every cubic form over K in more than $3t$ variables has a nontrivial zero in K. By virtue of Chevalley's result [Abh. Math. Sem. Hamburg. Univ. **11**, 73 (1935)] that every form over a finite field of degree n in more than n variables has a nontrivial zero in that field, this implies that if P is finite, then every cubic form over K in more than 9 variables has a nontrivial zero in K.

E. R. Kolchin (New York, N. Y.).

Pickert, Günter. Eine Normalform für endliche rein-inseparable Körpererweiterungen. Math. Z. **53**, 133–135 (1950).

This paper gives a much simpler proof of the existence and invariance of the author's normal form [same Z. **52**, 81–136 (1949); these Rev. **11**, 313], using methods of Dieudonné [Summa Brasil. Math. **2**, no. 1, 1–20 (1947); these Rev. **10**, 5]. *G. Whaples* (Bloomington, Ind.).

Nakayama, Tadasi. Halblineare Erweiterung des Satzes der Normalbasis und ihre Anwendung auf die Existenz der derivierten (differentialen) Basis. II. Proc. Japan Acad. **22**, nos. 1–4, 55–60 (1946).

The following theorem is proved: Let L/K be normal separable and F a subfield of L with $(L:F)\leqq(L:K)$. Then L contains an element whose $(L:K)$ conjugates with respect to K form a modul basis for L over F. A generalization (for the case of an unramified prime) of E. Noether's [J. Reine Angew. Math. **167**, 147–152 (1932)] theorem on normal bases is proved, which applies to the modules mentioned in the above theorem and the similar theorem proved in part I of this paper [same Proc. **21** (1945), 141–145 (1949); these Rev. **11**, 316]. If L/K is separable and an iterative higher derivation $\xi\rightarrow\xi^{(\nu)}$, $\nu=1, 2, \cdots$, in the sense of Hasse and Schmidt [J. Reine Angew. Math. **177**, 215–237 (1937)] is defined on L, then it is shown without any assumption about degrees (compare part I) that for some $\xi\epsilon L$ an independent basis for L/K can be chosen from the $\xi^{(\nu)}$. Finally, known results on the Klein form and function problems [see R. Brauer, Math. Ann. **110**, 473–500 (1934) and references given there] are generalized to semilinear representations, under the assumption that the field contains an infinite number of elements, and examples are given to show this restriction is necessary. *G. Whaples.*

Tannaka, Tadao, and Terada, Fumiyuki. A generalization of the principal ideal theorem. Proc. Japan Acad. **25**, no. 8, 7–8 (1949).
Terada, Fumiyuki. On a generalization of the principal ideal theorem. Tôhoku Math. J. (2) **1**, 229–269 (1950).
Tannaka, Tadao. Some remarks concerning principal ideal theorem. Tôhoku Math. J. (2) **1**, 270–278 (1950).
Tannaka, Tadao. An alternative proof of a generalized principal ideal theorem. Proc. Japan Acad. **25**, no. 11, 26–31 (1949).

Tannaka, several years ago, conjectured the following extension of the principal ideal theorem of class field theory. Theorem 1: Let Ω be a cyclic intermediate field in the absolute class field K/k; then every ambiguous class of Ω/k becomes principal in K. This is equivalent to theorem 2: Let G be a metabelian group with Abelian commutator subgroup G', H be an invariant subgroup of G with G/H cyclic, S a generator of G/H, and A an element of H with $ASA^{-1}S^{-1}\epsilon H'$; then the Verlagerung $V(A)=\prod TA\overline{TA}^{-1}$ from H to G' is the unit element of G. (Here T runs over a fixed representative system of H/G' and $\overline{TA}$ means the representative of the coset TAG'.)

The first of the above papers announces that Terada has succeeded in proving theorem 2 by the method of Furtwängler [Abh. Math. Sem. Hamburg. Univ. **7**, 14–36 (1930)], and that Tannaka, using theorem 2, has proved theorem 3: Let K/k be "Strahlklassenkörper" mod $\mathfrak{f}$, Ω/k a cyclic intermediate field, and C an ambiguous class mod $\mathfrak{m}$ in Ω, where $\mathfrak{m}=\max\{\mathfrak{f}(K/\Omega), \mathfrak{F}(\Omega/k)\}$, where $\mathfrak{F}(\Omega/k)$ is the so-called "Geschlechtermodul" for Ω/k; then C will be contained in the principal class mod $\mathfrak{F}(K/k)$ in K. The second paper contains Terada's proof. The computations are very long. The third contains Tannaka's proof of theorem 3, together with a modification of Iyanaga's [ibid. **10**, 349–357 (1934)] proof of the original principal ideal theorem. The fourth contains a much simpler proof, by Tannaka, of theorem 2 according to Iyanaga's method.
G. Whaples (Bloomington, Ind.).

Theory of Groups

Vázquez García, R., and Valle Flores, E. A relation between the cardinal number of a set and its possibility of being a group. Bol. Soc. Mat. Mexicana **5** (1948), 1–6 (1950). (Spanish)

If M is a nonempty set, and f is a binary, single-valued, associative operation defined in M, with respect to which M is closed, and which satisfies the right and left cancellation laws, then M is said to be a semigroup with respect to f. The authors show that if M is a set such that every semigroup definable over M is a group, then M is finite.
F. Bagemihl (Rochester, N. Y.).

Todd, J. A. On a conjecture in group-theory. J. London Math. Soc. **25**, 246 (1950).

If H_1 and H_2 are subgroups of a finite group G, and if the number of elements of each class of conjugates in G contained in H_1 is equal to the number in H_2, then Todd's example shows that H_1 and H_2 are not necessarily isomorphic as inferred by D. E. Littlewood. *G. de B. Robinson.*

Šapiro-Pyateckiĭ, I. I. On an asymptotic formula for the number of Abelian groups whose order does not exceed n. Mat. Sbornik N.S. **26(68)**, 479–486 (1950). (Russian)

Let $A(n)$ be the number of Abelian groups of order n and write

$$S(x)=\sum_{n\leqq x}A(n),\quad S(x;k,l)=\sum_{n\leqq x,\, n\equiv l\,(\mathrm{mod}\,k)}A(n).$$

It is shown that, for large x,

$$S(x)=\gamma x+O(x^{\frac{1}{2}+\epsilon}), \tag{1}$$

$$S(x;k,l)=\gamma x\frac{A(d)}{k}\prod_{p\mid k/d}\prod_{n=2}^{\infty}(1-p^{-n})+O(k^{\frac{1}{2}}x^{\frac{1}{2}+\epsilon}), \tag{2}$$

where $\gamma=\prod_{k=2}^{\infty}\zeta(k)$, $k\geqq2$, $d=(l,k)<k$. A similar result is given for $l=0$. To obtain these results the properties of the Dirichlet series $\sum\chi(n)A(n)n^{-s}=\prod_{r=1}^{\infty}L(rs,\chi^r)$ are considered for characters modulo k. The proof of (2) appears to be valid only for $d=1$, since otherwise it is not necessarily true that $S(x;k,l)=A(d)S(x/d;k/d,l/d)$. Similarly, the treatment of the case $l=0$ breaks down when k is not prime as it is not then possible to separate out the factors $A(k^n)$ from the sum considered. As mentioned by the author in a note added after submission the result (1) has already been obtained in a sharper form by Kendall and the reviewer [Quart. J. Math., Oxford Ser. (1) **18**, 197–208 (1947); these Rev. **9**, 226]. (There is an error in the main theorem. Landau's parameter g is 1 and not 0, which means that the error term must be multiplied by $\log x$ giving $S(x)=\alpha x-\beta x^{\frac{1}{2}}+O(x^{\frac{1}{3}}\log^2 x)$.) It should also be mentioned that the result (1) is originally due to Erdös and Szekeres [Acta Litt. Sci. Szeged **7**, 95–102 (1934)], a fact unknown also to the reviewer and Kendall until recently.
R. A. Rankin (Cambridge, England).

Abraham, G. Classes of the n-dimensional Lorentz group. Proc. Indian Acad. Sci., Sect. A. **28**, 87–93 (1948).

The author obtains canonical forms for n-dimensional Lorentz matrices, that is, $n\times n$ matrices which leave the form $x^1y^1-\cdots-x^ny^n$ invariant. The main results have been given by Wigner [Ann. of Math. (2) **40**, 149–204 (1939)] for the case $n=4$. *A. H. Taub* (Urbana, Ill.).

Igusa, Jun-ichi. On a property of commutators in the unitary group. Mem. Coll. Sci. Univ. Kyoto Ser. A. **26**, 45–49 (1950).

Let $U(n)$, $SU(n)$, be the unitary, special unitary groups, respectively, of degree n. It is proved that the mapping $f(X, Y)=XYX^{-1}Y^{-1}$ of $U(n)\times U(n)$ onto $SU(n)$ is open at a point $\{A, B\}$ if and only if the matrices A and B generate a subgroup of $U(n)$ which is irreducible as its own representation. This is proved by showing that the linear mapping $F(X, Y)=(A^{-1}XA-X)+(B^{-1}YB-Y)$ from the direct sum $L(n)+L(n)$ into $SL(n)$ is an onto-mapping if and only if A and B satisfy the same condition as above; here $L(n)$, $SL(n)$ denote the Lie algebras of $U(n)$, $SU(n)$, respectively. Now let $\mathfrak{G}$ be any discrete group and $G=\{\Phi\}$ the set of all unitary representations of degree n of $\mathfrak{G}$. The author introduces in G the weak topology defined by the functions $a(\Phi)=\Phi(a)$ for each $a\epsilon\mathfrak{G}$, so that G becomes a compact space. If $\mathfrak{G}$ is given by a finite number of generators and relations, then G can be considered to be a compact real algebraic variety. In particular, if $\mathfrak{G}$ is the Poincaré group of a closed orientable surface C of genus p, then the above yields a criterion when a point on G is multiple. Also, a formula for the dimension of G is given in terms of p and n. By identifying unitarily equivalent representations, a space V is obtained whose dimension is given. If C is the Riemann surface of an algebraic curve, then V gives the topological structure of the hyperjacobian variety [cf. A. Weil, J. Math. Pures Appl. (9) **17**, 47–87 (1938); Toyama, Proc. Imp. Acad. Tokyo **20**, 554–557, 558–560 (1944); these Rev. **7**, 429]. The paper concludes with the statement (given without proof) that the Poincaré group of V has (if $p\geqq 2$) as free generators the canonical paths on C (as imbedded in V) which generalizes a known result on the classical Jacobian variety ($n=1$). [Reviewer's note. It seems possible that the equivalence classes of irreducible unitary representations might form a more natural, though probably also more difficult, generalization of the Jacobian variety.]

F. I. Mautner (State College, Pa.).

Malcev, A. I. On semi-simple subgroups of Lie groups. Amer. Math. Soc. Translation no. **33**, 43 pp. (1950).

Translated from Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **8**, 143–174 (1944); these Rev. **6**, 146.

Cohen, L. W., and Goffman, Casper. On completeness and category in uniform space. Amer. J. Math. **72**, 752–756 (1950).

Results obtained previously concerning completeness in ordered Abelian groups are improved [Trans. Amer. Math. Soc. **66**, 65–74 (1949); **67**, 310–319 (1949); these Rev. **11**, 44, 324]. It is proved that an ordered Abelian group G is a-complete if and only if, for every isolated proper subgroup I of G, G/I is nondiscrete and is complete as a topological group (topology being introduced via the ordering).

E. Hewitt (Seattle, Wash.).

Cohen, L. W., and Goffman, Casper. On completeness in the sense of Archimedes. Amer. J. Math. **72**, 747–751 (1950).

A special class of spaces with uniform structure is defined, modelled closely upon the uniform structure possessed by certain ordered Abelian groups which the authors have studied elsewhere [see the preceding review]. A sufficient condition is established under which such spaces are of the second category, in a certain sense. *E. Hewitt.*

Edrei, Albert. On mappings of a uniform space onto itself. Trans. Amer. Math. Soc. **69**, 528–536 (1950).

Let $(E, \mathfrak{U})$ be a uniform space and let $H\subset E^E$. The following definitions are made: H is almost smooth provided that if $V\epsilon\mathfrak{U}$, then there exists a finite $G\subset H$ such that $h\epsilon H-G$ implies $hUx\subset Vhx$ ($x\epsilon E$) for some $U\epsilon\mathfrak{U}$; H is equally almost smooth in case U is independent of h (these notions generalize the notions of uniform continuity and uniform equicontinuity of a set of functions); H has scattered smoothness provided that if $V\epsilon\mathfrak{U}$, then there exist an infinite $G\subset H$ and $U\epsilon\mathfrak{U}$ such that $hUx\subset Vhx$ ($h\epsilon G$, $x\epsilon E$); H is iterative provided that if $G\subset H$ is infinite, then $kg_1=g_2$ for some $k\epsilon H$ and some $g_1, g_2\epsilon G$ with $g_1\neq g_2$; H is strongly iterative provided that if $G\subset H$ is infinite and if $h\epsilon H$, then $kh=g$ for some $k\epsilon H$ and some $g\epsilon G$ (for example, the iterates of a mapping form a strongly iterative semigroup). Let E^E be provided with its uniform-convergence uniformity and let $hE=E$ ($h\epsilon H$). The following results are typical. If E is compact and if H is a strongly iterative almost smooth Abelian semigroup with scattered smoothness, then the derived set H' is a group. If E is compact and if H is an infinite strongly iterative equally almost smooth semigroup, then H' is a uniformly equicontinuous group.

W. H. Gottschalk (Philadelphia, Pa.).

Riss, Jean. Applications dérivables d'un groupe dans un autre. C. R. Acad. Sci. Paris **230**, 2069–2071 (1950).

Let G be a locally compact Abelian topological group. The author has previously defined [same C. R. **227**, 664–666 (1948); these Rev. **10**, 429], for r any continuous homomorphism of the real line R into G, and f a complex valued function on G, the derivative of f at x in the direction r to be the ordinary derivative with respect to t at 0 of $f(x+r(t))$, and denoted it by $d_rf(x)$. Let E_1 denote the functions which have such a derivative for every x and r. The author now defines a mapping φ of R into G to be differentiable if $\hat{x}o\varphi$ is a differentiable function on R for every continuous character $\hat{x}$ of G. This note defines and discusses the differential $\delta\varphi$ of such a φ. The existence of $\delta\varphi$ is asserted in a theorem, and $\delta\varphi$ is a mapping which assigns to each real number t a continuous homomorphism r of R into G such that the derivative of $fo\varphi$ at t equals $((d_rf)o\varphi)(t)$, i.e., $\delta\varphi$ carries the unit tangent vector to the line at t into the tangent vector in the direction r at $\varphi(t)$. So $\delta\varphi$ is a mapping of $R\rightarrow R(G)$, where $R(G)$ is the space of continuous homomorphisms of R into G. Theorem 2 asserts that such a φ can be factored into a mapping ϕ: $R\rightarrow R(G)$, and the mapping: $r\rightarrow r(1)$, of $R(G)\rightarrow G$. Also, ϕ is unique up to an additive constant if one imposes continuity, and ϕ is called the integral of $\delta\varphi$ in $R(G)$. A mapping θ of $G\rightarrow G'$ (G' also being locally compact Abelian) is defined to be differentiable if $\hat{x}'o\theta$ is differentiable on G for each character $\hat{x}'$ of G'. Then $\delta\theta$ is defined in obvious fashion. A theorem asserts that if G is connected, then such a θ is determined to within a translation by $\delta\theta$. Other theorems are stated. *W. Ambrose.*

Kawada, Yukiyosi. On some properties of covering groups of a topological group. J. Math. Soc. Japan 1, 203–211 (1950).

The author investigates covering groups using the definitions of Chevalley [Theory of Lie Groups. I., Princeton University Press, 1946; these Rev. 7, 412]. If U is a symmetric neighborhood of the identity in a topological group G, there is defined a topological group $Gr(U)$ by the method of Schreier [Abh. Math. Sem. Hamburg. Univ. 4, 15–32 (1925); 5, 233–244 (1927)]. A necessary and sufficient condition that a connected, locally connected group be simply connected is that G and $Gr(U)$ be isomorphic for every connected symmetric U. If G is connected, locally connected, and locally simply connected, and if U is simply connected, then $Gr(U)$ is simply connected. If G is connected, locally connected, and has the first countability axiom, a topological group G_0^* is called a generalized universal covering of G provided (1) G_0^* is connected, locally connected, and simply connected, (2) there is a normal subgroup F_0^* such that $G_0^*/F_0^*=G$, (3) F_0^* is totally disconnected and there are arbitrarily small open subgroups of F_0^*. If $U_1 \supset U_2 \supset \cdots$ form a basis for neighborhoods of e, then the groups $Gr(U_i)$ form an inverse sequence having a limit G_0^{**}. If G_0^{**} is connected and locally connected it is a generalized covering group. Furthermore, if G has a generalized covering it must be isomorphic to G_0^{**}. *D. Montgomery* (Princeton, N. J.).

Markov, A. A. Three papers on topological groups: I. On the existence of periodic connected topological groups. II. On free topological groups. III. On unconditionally closed sets. Amer. Math. Soc. Translation no. 30, 120 pp. (1950).

Translated from Izvestiya Akad. Nauk SSSR. Ser. Mat. 8, 225–232 (1944); 9, 3–64 (1945); Mat. Sbornik (N.S.) 18(60), 3–28 (1946); these Rev. 7, 7, 412.

NUMBER THEORY

Gloden, A. Factorisation de nombres $N^{16}+1$. Euclides, Madrid 10, 157 (1950).

The factorization of $N^{16}+1$ is given completely for $N=2(1)13, 21, 27, 32$. *D. H. Lehmer* (Berkeley, Calif.).

Gloden, A. Une extension des systèmes multigrades. Euclides, Madrid 10, 289–290 (1950).

Three known theorems concerning equal sums of like powers are stated. *I. Niven* (Eugene, Ore.).

Jarden, Dov. On the numbers V_{6n} (n odd) in the sequence associated with Fibonacci's sequence. Riveon Lematematika 4, 38–40 (1950). (Hebrew. English summary)

The author discusses the factorization

$$V_{6m}=V_m(3+V_{2m}-5U_m)(3+V_{2m}+5U_m), \qquad m=2j+1,$$

where U is the nth term of the Fibonacci sequence $1, 1, 2, 3, 5, 8, \cdots$, $U_{n+1}=U_n+U_{n-1}$, and $V_n=U_{2n}/U_n$. The table of Kraitchik [Recherches sur la théorie des nombres, v. 1, Gauthier-Villars, Paris, 1924, p. 80], giving the decomposition of these factors into primes, is here reproduced and extended to $m=77$, most of the new entries being incompletely factorized. *D. H. Lehmer* (Berkeley, Calif.).

Majumdar, Kulendra N. On numbers which can be expressed by a given form. Proc. Nat. Inst. Sci. India 16, 99–100 (1950).

The author proves that for every k and for every sufficiently large x there is at least one integer between x and $x+Cx^{(1-k^{-1})^k}$ which is the sum of k positive integral kth powers. For $k=2$ this becomes a theorem of Bambah and Chowla [same Proc. 13, 101–103 (1947); these Rev. 9, 273]. This is a special case of a general theorem about integers of the form $\sum_{r=1}^{k}\alpha_r x_r^{m}$, the proof of which is presented. For the case $k=2$ the author states that for all $x \geqq 0$ there is always a sum of two squares between x and $x+(64x)^{\frac{1}{4}}+1$.
D. H. Lehmer (Berkeley, Calif.).

Rodeja F., E. G. Note on prime numbers. Gaceta Mat. (1) 1, 180–182 (1949). (Spanish)

The author proves that the positive integer $S \neq 4$ is a prime if and only if the product $\prod(r_1r_2-1)$, $r_1=1, \cdots, S-1$; $r_2=1, \cdots, S-1$; r_1, r_2 not both $=1$, is divisible by S^{S-1}.
H. W. Brinkmann (Swarthmore, Pa.).

Medgyessy, P. Eine geometrische Kennzeichnung der Primzahlen. Elemente der Math. 5, 114–115 (1950).

The author proves the following theorem: Denote by D_k the triangle with the vertices $(0, 0)$, $(n, k-1)$, (n, k). Let $n \neq 4$; then n is a prime if and only if all the triangles $D_2, D_3, \cdots, D_{n-1}$ contain the same number of lattice points in their interior. *P. Erdös* (Aberdeen).

Nicol, Hugh. Sieves of Eratosthenes. Nature 166, 565–566 (1950).

Dietrich, Verne E., and Rosenthal, Arthur. A remark about our note "Transcendence of factorial series with periodic coefficients." Proc. Amer. Math. Soc. 1, 825 (1950).

Cf. Bull. Amer. Math. Soc. 55, 954–956 (1949); these Rev. 11, 331.

Iwamoto, Yosikazu. A proof that π^2 is irrational. J. Osaka Inst. Sci. Tech. Part I. 1, 147–148 (1949).

The argument follows the reviewer's formulation [Bull. Amer. Math. Soc. 53, 509 (1947); these Rev. 9, 10] of the classical proof of the irrationality of π. A similar extension to π^2 has been done by Wachs [Bull. Sci. Math. (2) 73, 77–95 (1949); these Rev. 11, 418]. *I. Niven.*

Kraft, Émile. Essais et recherches sur la théorie des nombres. Revista Acad. Colombiana Ci. Exact. Fis. Nat. 7, 557–567 (1950).

The author considers the function

$$f(x)=\sum_{\lambda=1}^{\infty}(\sin \pi x/2\lambda)^{2(px)^q}$$

and notes that if x is an integer, $f(x)$ approximates the number of odd divisors of x with an error which tends to zero for $q>2$. For $q=2$ the error depends upon p and is essentially a theta function. A graph of $f(x)$ for $x \leqq 50$ is given. The reader is advised to be on the lookout for misleading typographical errata. *D. H. Lehmer.*

Niven, Ivan. The iteration of certain arithmetic functions. Canadian J. Math. 2, 406–408 (1950).

This paper is concerned with properties of the following arithmetic functions: (a) for $n \geqq 3$ let $C(n)$ equal the integer j such that the jth iterate of the Euler φ-function equals two

(i.e., $\varphi^{(j)}(n)=2$), $C(2)=C(1)=0$; and (b) let $g(n)$ equal the least positive integer j such that $\lambda^{(j)}(n)=1$, where $\lambda^{(j)}(n)$ is the jth iterate of the function $\lambda(n)$ which is in turn the least exponent such that $a^{\lambda(n)}\equiv 1 \pmod n$. The main results are: (1) if $(a,b)=1$, then $g(ab)=\max\{g(a),g(b)\}$;

$$\text{(2)}\quad \limsup\{C(n+1)-C(n)\}=\limsup\{g(n+1)-g(n)\}=\limsup\{C(n)-g(n)\}=\infty;$$

$$\text{(3)}\quad \liminf\{C(n+1)-C(n)\}=\liminf\{g(n+1)-g(n)\}=-\infty;$$

and

$$\text{(4)}\quad \liminf\{C(n)-g(n)\}=-1.$$

H. N. Shapiro (New York, N. Y.).

Anfert'eva, E. A. Summation formulas containing special numerical functions. Mat. Sbornik N.S. **27**(69), 69–84 (1950). (Russian)

Various summation formulae are obtained by utilizing the functional equation for $\varphi_\nu(s)=\zeta(s+\nu)\zeta(s-\nu)$ (obtained from that of the ζ function). The method used is that given by Ferrar [Compositio Math. **4**, 394–405 (1937)] for obtaining such summation formulae from a rather general class of functions possessing a functional equation.

H. N. Shapiro (New York, N. Y.).

Mahler, Kurt. A correction. Compositio Math. **8**, 112 (1950).

The author corrects a minor error in a previous paper [same journal **2**, 259–275 (1935)] and shows that the only zeros of the dyadic logarithm function are ± 1.

L. Schoenfeld (Urbana, Ill.).

Maler, K. [Mahler]. On a theorem of Dyson. Mat. Sbornik N.S. **26**(68), 457–462 (1950). (Russian)

Let m and n be positive integers with $n>m$; let x_i, y_i be two independent sets of n integer variables; let f_j be n linear forms in the x_i, and let g_i be n linear forms with integer coefficients and determinant ± 1 in the y_i. Let $\sum_{j=1}^n f_j g_j=\sum_{i,k=1}^n e_{ik}x_i y_k$, where the e_{ik} are integers. Let a_j be n positive real numbers satisfying $a_j\geqq 1$ for $j\leqq m$ and $a_j\leqq 1$ for $j>m$; let z be the unique number such that $z\prod_{j=1}^m \max(1, z^{-1}a_j)\prod_{j=m+1}^n \min(1, z^{-1}a_j)=1$. The following theorem is proved. If a set of integers x_i, not all zero, exists such that $f_j\leqq a_j^{-1}$, $j=1,\cdots,(n-1)$, $f_n=a_n^{-1}$, then a set of integers y_i, not all zero, exists such that $g_j<\max(1, z^{-1}a_j)$, $j=1,\cdots,m$; $g_j\leqq\min(1, z^{-1}a_j)$, $j=m+1,\cdots,n-1$; $g_n\leqq(n-1)\min(1, z^{-1}a_n)$. This theorem is a generalization of one proved by a more complicated method by the reviewer [Proc. London Math. Soc. (2) **49**, 409–420 (1947); these Rev. **9**, 271].

F. J. Dyson (Birmingham).

Bellman, Richard. Generalized Eisenstein series and non-analytic automorphic functions. Proc. Nat. Acad. Sci. U. S. A. **36**, 356–359 (1950).

The author introduces a new class of "nonanalytic automorphic functions" [cf. Maass, Math. Ann. **121**, 141–183 (1949); these Rev. **11**, 163]. These new functions are obtained from the classical theta-functions, and the author uses the theta transformation formula to expand the new functions into generalized Eisenstein series. The associated Dirichlet series are the Epstein ζ-functions. The author indicates two generalizations of his result: one, to the Siegel modular functions; the other, to the function $f_k(x)=\sum_{n=1}^\infty d_k(n)V_k(n^2x^2)$, where V_k is the Voronoi function defined by $\int_0^\infty V_k(x)x^{s-1}dx=\{\Gamma(s)\}^k$. Proofs and details are promised in a subsequent paper.

J. Lehner.

Maass, Hans. Modulformen zweiten Grades und Dirichletreihen. Math. Ann. **122**, 90–108 (1950).

The author develops a Hecke theory of Dirichlet series satisfying functional equations for a particular case ($n=2$) of Siegel's matrix modular forms. The method is analogous to that by which Hecke treated modular forms on the Hilbert modular group (double ϑ-series) and extracted from them his $\zeta(s,\lambda)$ functions for a real quadratic field. Siegel's modular forms are functions $g(Z)$ of three complex variables

$$z_0, z_1, z_2;\quad Z=\begin{pmatrix} z_0 & z_1\\ z_1 & z_2\end{pmatrix}=\begin{pmatrix} x_0 & x_1\\ x_1 & x_2\end{pmatrix}+i\begin{pmatrix} y_0 & y_1\\ y_1 & y_2\end{pmatrix}=X+iY,$$

analytic in the space $Y>0$ and having the automorphic property $g(Z')=|CZ+D|g(Z)$ for every modular substitution of degree 2, i.e., $Z'=(AZ+B)(CZ+D)^{-1}$, where $AB'=BA'$, $CD'=DC'$, $AD'-BC'=E$ (unit matrix), A, B, C, D are quadratic matrices with integer elements; $-k$ is the "dimension" of g. The function $g(Z)$ is expanded in a Fourier series (1) $\sum a(Z)\exp\{2\pi i\cdot\operatorname{trace}(TZ)\}$, summed over all nonnegative symmetric

$$T=\begin{pmatrix} t_0 & t_1\\ t_1 & t_2\end{pmatrix}.$$

To this we associate through the Mellin transform the Dirichlet series $\xi_1+\xi_2$, where

$$\xi_\nu(s,\tau;g)=(2\pi)^{-2s}\Gamma(2s)\sum a(T)(\operatorname{trace}(TY_1))^{-2s}$$

over T of rank ν, where we have set $X=0$, parameterized

$$Y=uY_1,\quad Y_1=y^{-1}\begin{pmatrix} x^2+y^2 & x\\ x & 1\end{pmatrix},\quad u>0, y>0,$$

and put $\tau=x+iy$.

In Hecke's theory, we subtract off the constant term of the Fourier series (1); the remainder, when subjected to the Mellin transform, yields a series whose Fourier coefficients are the associated Dirichlet series; under certain conditions, the process is reversible: the modular form and the Dirichlet series determine each other uniquely. In the present paper the author by analogy subtracts off from (1) not only the constant term but all terms in which T is of rank 1; i.e., he considers $\xi_2(s,\tau;g)$ and subjects it to a Fourier analysis, obtaining in this way the desired Dirichlet series. Calling $R(s;e,g)$ the Fourier coefficient of ξ_2, he obtains R as a Dirichlet series multiplied by Γ-factors, as in Hecke's theory; by means of another representation of R he shows that R is a meromorphic function of s in the whole plane having a finite number of simple poles. Furthermore, R satisfies the functional equation $R(k-s;e,g)=(-1)^kR(s;e,g)$.

The Fourier analysis of ξ_2 is not made with respect to the exponential function, as in Hecke's case, but in terms of certain "automorphic wave-functions" which the author introduced in an earlier paper [same Ann. **121**, 141–183 (1949); these Rev. **11**, 163]. These are the functions $e(\tau)$ in $R(s;e,g)$. They are the analogues of the Hecke "Grössencharakter" λ^n in Hecke's Dirichlet series $\sum_{(\mu)}\lambda^n(\mu)/|N(\mu)|^s$ extended over certain principal ideals in the real quadratic field. Whether conversely, the Dirichlet series for R determines the modular form g is not settled in this paper.

J. Lehner (Philadelphia, Pa.).

***Müller, Oskar. Über das Minimum des Produktes dreier ternärer linearer Formen.** Thesis, University of Zürich, 1947. 36 pp.

This thesis consists of a geometric proof of a theorem obtained arithmetically by Davenport [Proc. London Math.

Soc. (2) **44**, 412–431 (1938)], namely: If ξ, η, ζ are three linear forms in three variables with real coefficients and determinant D, then there exist rational integral values of the variables not all zero such that $|\xi\eta\zeta| \leqq |D|/7$, the equality holding only when $\xi\eta\zeta$ is equivalent to a multiple of $x^3+y^3+z^3-x^2y-x^2z-2xy^2-2xz^2+3xyz+3y^2z-4yz^2$. Brief applications are given. *B. W. Jones* (Boulder, Colo.).

Godwin, H. J. On the product of five homogeneous linear forms. J. London Math. Soc. **25**, 331–339 (1950).

Let $L_1, \cdots, L_5$ be five homogeneous linear forms, with real coefficients, in five variables $u_1, \cdots, u_5$. The determinant of the forms is taken, without loss of generality, to be 1. The author proves that there exist integral values, not all zero, of $u_1, \cdots, u_5$ for which $|L_1 \cdots L_5| < (57.02)^{-1}$. The method is based on that devised by the reviewer for the product of three linear forms [same J. **16**, 98–101 (1941); these Rev. **3**, 70]. The essential lemma is that if $|(n-\alpha)(n-\beta)(n-\gamma)(n-\delta)(n-\epsilon)| \geqq 1$ for all integers n, then $\sum(\alpha-\beta)^2 > 35.71$. The proof of this depends, as one might expect, on a somewhat elaborate division into cases, with a variety of treatments for the various cases.
H. Davenport (London).

Inkeri, K. On the Minkowski constant in the theory of binary quadratic forms. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **66**, 35 pp. (1950).

Let $f(x, y) = ax^2+bxy+by^2$ be a real quadratic form with positive discriminant $d = b^2-4ac$. Let $\mathfrak{M}$ be the least number with the property that for any real x_0, y_0, real X, Y exist with $X \equiv x_0 \pmod 1$, $Y \equiv y_0 \pmod 1$, and $|f(X, Y)| \leqq \mathfrak{M}$. Minkowski proved $\mathfrak{M} < \frac{1}{4}\sqrt{d}$. The purpose of this paper is either to determine $\mathfrak{M}$ exactly or to give estimates for it for special classes of forms. Forms representing 0 are excluded. Let $D = d/4a^2$. It is assumed $\frac{1}{4} < D$, otherwise $f(x, y)$ is replaced by an equivalent form with $\frac{1}{4} < D$. Let r be a real number with $r \equiv -b/2a \pmod 1$. The principal results of the paper consist of a number of different bounds for $\mathfrak{M}$ in the case where $D < 1$ which depend on r, a, and D. Typical of these is the following. For $r \geqq 0$, $D \geqq r^2+r$, $\mathfrak{M} \leqq C|a|$, where C is

$$\max \{\tfrac{1}{4}D, \tfrac{1}{4}[(2D-r^2)^{\frac{1}{2}}-r]^2, \tfrac{1}{4}[1-(D-r^2-r)^2/D]\}.$$

These results for forms with appropriate values of r are improvements of Davenport's result that $\mathfrak{M} \leqq \frac{1}{4}|a|$ [Nederl. Akad. Wetensch., Proc. **49**, 815–821 = Indagationes Math. **8**, 518–524 (1946); these Rev. **8**, 444]. Davenport's result is a consequence of one of the author's theorems which also yields a proof of the following result of Heinhold [Math. Z. **44**, 659–688 (1939)]. For a squarefree integer m let $f(x, y) = x^2-my^2$ or $x^2+xy+\frac{1}{4}(1-m)y^2$ according as $m \not\equiv 1 \pmod 4$ or $m \equiv 1 \pmod 4$. If p, p_1, q are positive integers with $m = q^2+p = (q+1)^2-p_1$ or $m = (2q+1)^2+4p = (2q+3)^2-4p_1$ according as $m \not\equiv 1 \pmod 4$ or $m \equiv 1 \pmod 4$, then

$$\mathfrak{M} \leqq \tfrac{1}{2}\max(p, p_1) = M_0.$$

For $m \equiv 1 \pmod 4$ it is shown that $p = p_1$ implies $\mathfrak{M} = M_0$, and conversely. For the remaining m a number of criteria are given which imply that $\mathfrak{M} = M_0$. Criteria are also given which insure that $\mathfrak{M} < M_0$. The author applies his results to the forms defined by the squarefree integers m with $m \leqq 101$. In most cases where $\mathfrak{M} < M_0$ a smaller bound for $\mathfrak{M}$ is given than M_0. For $m = 7$ and $m = 69$ simple computations give the exact value of $\mathfrak{M}$ ($< M_0$). The first of these values was given earlier by Varnivides [Proc. Roy. Soc. London. Ser. A. **197**, 256–268 (1949); these Rev. **10**, 682], the second is new. *D. Derry* (Vancouver, B. C.).

Rédei, L. Endlich-projektivgeometrisches Analogon des Minkowskischen Fundamentalsatzes. Acta Math. **84**, 155–158 (1950).

The author points out that Minkowski's fundamental theorem has the following corollary. Suppose that $m_1, \cdots, m_k$ are positive integers and that $\mathfrak{K}$ is a convex body in n-dimensional Euclidean space symmetrical in the origin and having volume not less than $2^n m_1 \cdots m_k$. Let $L_i(x_1, \cdots, x_n)$, $i = 1, \cdots, k$, be homogeneous linear forms with integral coefficients (or, more generally, functions on ordered n-tuples of integers taking integral values and having the property that $L_i(x_1, \cdots, x_n) \equiv L_i(y_1, \cdots, y_n) \pmod{m_i}$ implies $L_i(x_1-y_1, \cdots, x_n-y_n) \equiv 0 \pmod{m_i}$). Then there exists a nonzero lattice point $(x_1, \cdots, x_n)$ in $\mathfrak{K}$ such that $L_i(x_1, \cdots, x_n) \equiv 0 \pmod{m_i}$, $i = 1, \cdots, k$. The title of the paper refers to the special case of the preceding result in which $0 \leqq k \leqq n-1$, $m_1, \cdots, m_k$ are all equal to the same prime number p, and $\mathfrak{K}$ contains no nonzero lattice point all coordinates of which are divisible by p. The further specialization in which $k = 1$ and $n = 2$ is a generalization of a theorem of Thue [Christiania Vid. Selsk. Forh. **1902**, no. 7]. In another paper [Nieuw Arch. Wiskunde (2) **23**, 150–162 (1950); these Rev. **11**, 417] the author has given some applications of this case to power residues mod p.
P. T. Bateman (Urbana, Ill.).

Parry, C. J. The $\mathfrak{p}$-adic generalisation of the Thue-Siegel theorem. Acta Math. **83**, 1–100 (1950).

In der umfangreichen Abhandlung wird im wesentlichen ein Satz von Mahler verallgemeinert und aus dem neuen Ergebnis werden einige Folgerungen gezogen. Der Mahlersche Satz [Math. Ann. **107**, 691–730 (1933); **108**, 37–55 (1933)] gibt eine Aussage über die Approximation algebraischer Zahlen durch rationale bei archimedischer, wie auch nicht-archimedischer Bewertung, und ist in diesem Sinne als Übertragung des bekannten Thue-Siegelschen Satzes [Siegel, Math. Z. **10**, 173–213 (1921)] aufzufassen. Er lautet: Seien $P_1, \cdots, P_\sigma$ (mit $\sigma \geqq 0$) verschiedene natürliche Primzahlen und $\xi_0, \cdots, \xi_\sigma$ reelle P_1-adische, $\cdots$, P_σ-adische Wurzeln eines irreduziblen Polynoms $f(x)$ vom Grade $m \geqq 3$ mit ganzrationalen Koeffizienten. Sei $\alpha = \min_{s=1,\cdots,m-1}(m/(s+1)+s)$ und β eine Zahl mit $\alpha < \beta \leqq m$, ferner c eine positive Konstante. Dann ist die Anzahl der Lösungen der Ungleichung

$$\min\left(1, \left|\frac{p}{q}-\xi_0\right|\right)\prod_{k=1}^{\sigma}\min(1, |p-\xi_k q|_{P_k}) \leqq c\max(|p|, |q|)^{-\beta}$$

in Paaren relativ primer ganzrationaler Zahlen p, q nicht grösser als $c_0 2^{\beta(1+\epsilon_0)\sigma/(\beta-\alpha)}$, wobei ϵ_0 eine positive Zahl und c_0 eine Konstante, die nur von ϵ_0, β, und $f(x)$, nicht aber von $P_1, \cdots, P_\sigma$ abhängt, ist ($|p-\xi_k q|_{P_k}$ berechne die P_k-adische Bewertung von $(p-\xi_k q)$). Der Verfasser verallgemeinert diesen Satz der Approximation durch rationale Zahlen nun auf die Annäherung algebraischer Zahlen durch algebraische Zahlen bei archimedischer und nicht-archimedischer Bewertung. Sein Hauptergebnis lautet: "Let $f(x)$ be a polynomial of degree $m \geqq 2$ with integral coefficients from $\mathfrak{K}$ and a nonzero discriminant. Let $\mathfrak{q}_1, \mathfrak{q}_2, \cdots, \mathfrak{q}_\rho$, where $0 \leqq \rho \leqq r_1+r_2$, be ρ of the r_1+r_2 infinite prime ideals corresponding to the r_1 real and r_2 pairs of conjugate imaginary fields conjugate to $\mathfrak{K}$, and let $\mathfrak{r}_1, \mathfrak{r}_2, \cdots, \mathfrak{r}_\sigma$, where $\sigma \geqq 0$, be σ different finite prime ideals of $\mathfrak{K}$. Let

$h_{k\delta}$ $(k=1, 2, \cdots, \sigma; \delta=1, 2, \cdots, G(\mathfrak{r}_k))$ be a natural number not greater than h^2. Let $\xi_{j\gamma}$ $(j=1, 2, \cdots, \rho; \gamma=1, G(\mathfrak{q}_j))$ be a real or complex root of $f(x)$ and $\eta_{k\delta\tau}$ $(k=1, 2, \cdots, \sigma; \delta=1, 2, \cdots, G(\mathfrak{r}_k); \tau=1, 2, \cdots, h_{k\delta})$ an $\mathfrak{r}_k$-adic root of $f(x)$, and let t be the total number of these roots. Let c and ϵ_0 be two positive numbers and α and β two numbers such that $\alpha=\min_{s=1,\cdots,m-1}(m/(s+1)+s)$ and $\beta>\alpha$. Then the number of different algebraic numbers λ of degree h (or any divisor of h) over $\mathfrak{R}$, lying in the perfect $\mathfrak{r}_1$-adic, $\mathfrak{r}_2$-adic, $\cdots$, $\mathfrak{r}_\sigma$-adic extensions of $\mathfrak{R}$ and satisfying the inequality

$$\prod_{j=1}^{\rho}\prod_{\gamma=1}^{G(\mathfrak{q}_j)} \min(1, |\lambda-\xi_{j\gamma}|_{\mathfrak{q}_{j\gamma}}) \prod_{k=1}^{\sigma}\prod_{\delta=1}^{G(\mathfrak{r}_k)}\prod_{\tau=1}^{h_{k\delta}} \min(1, |\lambda-\eta_{k\delta\tau}|_{\mathfrak{r}_k}) \leqq c\Lambda^{-h\beta}$$

is not greater than $k_0 2^{\beta(1+\epsilon_0)t/(\beta-\alpha)}$, where k_0 is a constant depending only on ϵ_0, β, c, $\mathfrak{R}$, $f(x)$, and h, and not on the number and choice of the roots to which approximation is made or on the corresponding ideals." *T. Schneider.*

Korobov, N. M. Concerning some questions of uniform distribution. Izvestiya Akad. Nauk SSSR. Ser. Mat. 14, 215–238 (1950). (Russian)

If, for each choice of the positive integer s and of the integers $m_1, \cdots, m_s$, not all 0, the function

$$m_1f(x+1)+\cdots+m_sf(x+s)$$

is uniformly distributed, then $f(x)$ is said to be completely uniformly distributed. Polynomials are not completely uniformly distributed although some of them are uniformly distributed. Using estimates of exponential sums due to Vinogradov and his followers, the author establishes the existence of a class of completely uniformly distributed functions $f(x)$ such that for suitable positive λ_1 and λ_2, $f(x)=o(x^{\lambda_1 \log\log x})$, $f(x)\neq o(x^{\lambda_2 \log\log x})$. Now suppose that $b_k=\pm1$, $\lambda>3$, $w(k)\geqq k^\lambda$, and $1+1/k\leqq w(k+1)/w(k)\leqq k$ for all sufficiently large k; using the previous result, the author shows that $f(x)=\sum_{k=0}^\infty b_k e^{-w(k)}x^k$ is uniformly distributed. By taking $b_k=1$ and $w(k)=k^{1+2/(1-\epsilon)}$, the author shows that there is a function of this kind for which $f(x)>\exp\{(\log x)^{3/2-\epsilon}\}$.

The author next derives a necessary and sufficient condition on α for the function $\alpha q_1\cdots q_x$ to be uniformly distributed under the assumption that the q_x's are integers greater than 1 and tend to infinity. He also obtains two sufficient conditions on α which insure the uniform distribution of αq^x if $q\geqq2$. Taking $q_x=x+1$, it follows from the first of these results that $\alpha\cdot x!$ is uniformly distributed if $\alpha=\sum_{k=1}^\infty[k^{1+\lambda}]/k!$ and $0<\lambda<1$; other specializations of these results are also given. While the uniform distribution of these functions for almost all values of α (in the sense of measure 0) has been known since Weyl's work [Math. Ann. 77, 313–352 (1916)], not a single value of α has been known for which these functions actually are uniformly distributed.

L. Schoenfeld (Urbana, Ill.).

Korobov, N. M. Normal periodic systems and a question on the sums of fractional parts. Uspehi Matem. Nauk (N.S.) 5, no. 3(37), 135–136 (1950). (Russian)

The author gives a brief indication of the proofs of the following results, the second of which depends on a result proved in the paper reviewed above. If $\epsilon(p)\to0^+$ as $p\to\infty$, then there exists an α for which αq^x is uniformly distributed and such that $S(p)\neq o(p\epsilon(p))$, where

$$S(p)=\sum_{x=1}^{p}(\alpha q^x-[\alpha q^x])-\tfrac{1}{2}p;$$

thus, the error term $o(p)$, which is a consequence of the uniform distribution of αq^x, cannot be improved for all α. If $\phi(p)\to\infty$ as $p\to\infty$, then there exists an α for which αq^x is uniformly distributed and such that $S(p)=o(\phi(p))$.

L. Schoenfeld (Urbana, Ill.).

ANALYSIS

Kinokuniya, Yoshio. Mean-value theorem and distribution densities. Kōdai Math. Sem. Rep. 1950, 53–55 (1950).

Let f_1 be a bounded positive Lebesgue measurable function of the real variable x, with integral 1 over the x-axis, whose reciprocal is bounded in every finite interval. Define $f_2, f_3, \cdots$ successively by $f_{n+1}(x)=\int_{-\infty}^{\infty} f_n(x-y)f(y)dy$. Then it is shown that $\lim_{n\to\infty} f_n(x)=0$ uniformly in x. If f_1 is differentiable, this result is used to obtain information on the size of θ in the law of the mean as applied to f_1 with a large increment in x. *J. L. Doob* (Urbana, Ill.).

Müller, Max. Bemerkungen zur Fehlertheorie. Math. Z. 52, 735–749 (1950).

Let there be given an infinite sequence of numbers $\epsilon_1, \epsilon_2, \cdots, \epsilon_\nu, \cdots$. Let $A_n(\epsilon_\nu; \alpha, \beta)$ denote the number of ϵ's in the finite sequence $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$ which satisfy the inequality $\alpha\leqq\epsilon_\nu\leqq\beta$. If there exists a nonnegative R-integrable function $\varphi(\epsilon)$ such that, for every interval (α, β), $\lim_{n\to\infty} A_n(\epsilon_\nu; \alpha, \beta)/n=\int_\alpha^\beta\varphi(\epsilon)d\epsilon$, the function $\varphi(\epsilon)$ is called the density of the sequence $\{\epsilon_\nu\}$. The author raises the inverse question. Given a function $\varphi(\epsilon)$, does a corresponding sequence $\{\epsilon_\nu\}$ exist? Under liberal hypotheses he establishes the existence of the sequence $\{\epsilon_\nu\}$, shows by a counter example that some limitation is necessary, and proves several additional theorems. *W. E. Milne.*

Rényi, Alfréd. On Newton's method of approximation. Mat. Lapok 1, 278–293 (1950). (Hungarian. Russian and English summaries)

In applying Newton's method it is customary to restrict the first approximation to a small interval containing only one root. The author uses a completely arbitrary point as the first approximation and examines the convergence of the iteration. In this case the method does not necessarily lead to a convergent sequence. Starting points for which the sequence is not convergent are termed singular. The following theorem is proved: Let $f(x)$ be a function with continuous first and second derivatives such that $f''(x)$ is monotone increasing for all x, and assume also that $f(x)=0$ has exactly three real roots A_1, A_2, A_3. The set of all singular points is then enumerable. Furthermore, there exists for each ϵ an interval I_ϵ of length less than ϵ and three points a_1, a_2, a_3 in I_ϵ such that the iteration converges to the root A_i when a_i, $i=1, 2, 3$, is used as the first approximation. In addition the author states the conjecture that the set of singular points is enumerable if $f(x)$ is a polynomial having only real roots. He also raises the question whether it is possible to find a polynomial with at least one real root which has the property that the set of its singular points contains an interval.

E. Lukacs (Washington, D. C.).

Kakehashi, Tetsujiro. Notes on the convergence of interpolations. J. Osaka Inst. Sci. Tech. Part I. 1, 83–85 (1949).

Referring to an estimate of the remainder of Lagrange interpolation given previously by the author [same vol., 5–11 (1949); these Rev. 11, 356], the following result is proved. We choose the roots of the Jacobi polynomial $P_{n+1}(\alpha, \alpha, x)$ as points of interpolation, $\alpha > -1$. Let the given function $f(x)$ satisfy the Lipschitz condition $|f^{(p)}(x_1) - f^{(p)}(x_2)| < K|x_1 - x_2|^a$, $0 < a \leqq 1$. A sufficient condition for the uniform convergence of the Lagrange interpolation in $(-1, 1)$ is the following: $p + a > 1 + \alpha + \frac{1}{2}\alpha^2$.

G. Szegö (Stanford University, Calif.).

Reiz, A. On a special case of a quadrature formula of Christoffel. Math. Tables and Other Aids to Computation 4, 181–185 (1950).

The mechanical quadrature formula of the Gauss-Jacobi type is studied, associated with the weight function $(x^2+a^2)^{-1}$, $a>0$, in the interval $(-1, +1)$. The corresponding orthogonal polynomials can be written in the form $P_n(x) - \lambda P_{n-2}(x)$, where $P_n(x)$ is Legendre's polynomial and $\lambda = Q_n(ia)/Q_{n-2}(ia)$; here $Q_n(z)$ denotes the Legendre function of the second kind.

G. Szegö (Stanford University, Calif.).

Tagamlitzki, J. Über Zahlenfolgen, die gewissen Ungleichungen genügen. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. 43, 193–237 (1947). (Bulgarian. German summary)

[Volume number misprinted 42 on title page.] Proof of the following and related theorems on sequences and functions. If $f(x)$ satisfies, for each $k=0, 1, 2, \cdots$ and $x>a$, the inequality $|f^{(k)}(x)| \leqq Ae^{-x}$, then there is a constant C such that $f(x) = Ce^{-x}$ for $x>a$. *R. P. Agnew* (Ithaca, N. Y.).

Tagamlitzki, Y. Recherches sur une classe de fonctions. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. (Math. Phys.) 44, 317–356 (1948). (Bulgarian. French summary)

For the main results of this paper see a series of notes by the author [Doklady Akad. Nauk SSSR (N.S.) 57, 875–878 (1947); 58, 197–200 (1947); C. R. Acad. Sci. Paris 223, 940–942 (1946); 225, 976–978, 1053–1055 (1947); these Rev. 9, 237; 8, 259; 9, 182, 280]. The results of the first two notes are given new proofs and appear in generalized form. The author begins by considering the class $K(a)$ of functions having derivatives of all orders on $[a, \infty)$ and satisfying $f^{(k)}(x) = o(x^{-k})$ as $x \to \infty$, and introducing a partial order by writing $f_1(x) \subset f_2(x)$ if $f_1 - f_2$ is completely monotonic for $x>a$. If $\varphi \supset f$ for every f of a subset of $K(a)$, φ is called a majorant for that subset; if a subset has a majorant, it has a smallest majorant. A series of theorems establish the expected properties of $K(a)$ as a partially ordered linear space. As an example of the author's applications, we sketch his proof that $f(x)$ is represented by a Dirichlet series with given exponents λ_ν if there is a $g(x)$ with the same exponents and nonnegative coefficients such that $|f^{(k)}(x)| \leqq (-1)^k g^{(k)}(x)$, $k=0, 1, 2, \cdots$ [cf. the first note cited above and Boas, C. R. Acad. Sci. Paris 224, 1683–1685 (1947); these Rev. 8, 569]. Let $\varphi(x) = g(x) + f(x)$; then $0 \subset \varphi(x) \subset 2g(x)$. Let $\theta_n(x)$ be the common majorant of $\varphi(x)$ and $2s_n(x)$, where s_n consists of the first n terms of $g(x)$; this majorant exists in virtue of the author's theorem that two functions with a common minorant have a common majorant. It follows from properties of the partial ordering that $0 \subset \theta_n(x) - \theta_{n-1}(x) \subset 2B_n e^{-\lambda_n x}$, $n=1, 2, 3, \cdots$; $\theta_0(x) = 0$. Hence the theorem is reduced to the case where $g(x)$ consists of a single term [see the third of the author's notes cited above]. A similar argument applies to Laplace integrals and eventually leads to a new proof of the Bernstein-Widder representation of completely monotonic functions.

R. P. Boas, Jr. (Evanston, Ill.).

Bernštein, S. N. On some properties of cyclically monotonic functions. Izvestiya Akad. Nauk SSSR. Ser. Mat. 14, 381–404 (1950). (Russian)

This paper contains proofs of a number of results previously stated without proof by the author [Atti Congresso Int. Mat. Bologna 1928, vol. 2, pp. 267–275 (1930)]. He calls $f(x)$ cyclically monotonic on (a, b) if $f(x)$ and its derivatives are each of constant sign and $f^{(k)}(x)f^{(k+2)}(x)$ is always nonpositive. Let $S_m(x)$ and $C_m(x)$ be the polynomials with leading term $x^m/m!$, such that the even derivatives of S_m and the odd derivatives of C_m vanish at 0, while the odd derivatives of S_m and the even derivatives of C_m vanish at 1 (for orders less than m). The author investigates properties of and relations among these polynomials and shows (theorem 1) that they deviate least from zero among all cyclically monotonic polynomials of degree m with the same leading term. From this he derives theorem 2, that a cyclically monotonic function on $(0, 1)$ coincides on $(0, 1)$ with an entire function of exponential type at most $\frac{1}{2}\pi$; and theorem 3, that a function is an entire function of exponential type at most p if and only if it can be represented on any interval of length less than $\frac{1}{2}\pi/p$ as the difference of two cyclically monotonic functions, but not so represented on some interval of greater length. There are numerous other remarks on cyclically monotonic functions. Theorem 4 determines, by means of C_m and S_m, the best upper bound for a polynomial $x^m/m! + \cdots$ such that it and its first $m-1$ derivatives vanish at least once in $0 \leqq x \leqq 1$. Theorem 5 states that if $f(x)$ and its first n derivatives each vanish in $0 \leqq x \leqq 1$, and $\max |f(x)| = 1$ on $(0, 1)$, then $\max |f^{(m)}(x)| \geqq 1/L_m \sim \frac{1}{4}\pi(\frac{1}{2}\pi)^m$ for $m \leqq n+1$; hence if $f(z)$ is entire and of type less than $\frac{1}{2}\pi$ and if $f(x)$ and each derivative vanish somewhere in $(0, 1)$, then $f(x) \equiv 0$. A final section deals briefly with generalizations to less restricted types of regularly monotonic functions.

Reviewer's remarks. The author's paper cited at the beginning of this review has been overlooked by several other writers who have rediscovered some of his results. A shorter proof of a sharper form of theorem 2 was found by the reviewer [Bull. Amer. Math. Soc. 47, 750–754 (1941); these Rev. 3, 77]. The corollary to theorem 5 was obtained by Schoenberg [Trans. Amer. Math. Soc. 40, 12–23 (1936); see also S. S. Macintyre, ibid. 67, 241–251 (1949); these Rev. 11, 340]. The polynomials C_m and S_m were investigated by J. M. Whittaker [Proc. London Math. Soc. (2) 36, 451–469 (1933)], who discussed their expansion properties.

R. P. Boas, Jr. (Evanston, Ill.).

Theory of Sets, Theory of Functions of Real Variables

Erdös, P., and Rado, R. A combinatorial theorem. J. London Math. Soc. 25, 249–255 (1950).

Ramsey [Proc. London Math. Soc. (2) 30, 264–286 (1929)] proved the following theorem: If Δ is an arbitrary distribution of all sets of n positive integers into a finite

number of classes, then there exists an infinite set M of positive integers such that all sets of n numbers of M belong to the same class of Δ. Let $k, \nu_1, \cdots, \nu_k$ be integers, $0 \leqq k \leqq n$, $0 < \nu_1 < \cdots < \nu_k \leqq n$. The canonical distribution $\Delta^{(k)}_{\nu_1, \ldots, \nu_k}$ of all sets of n positive integers is defined as one in which two sets $\{a_1, \cdots, a_n\}$, $\{b_1, \cdots, b_n\}$ with $a_1 < \cdots < a_n$, $b_1 < \cdots < b_n$ belong to the same class if and only if $a_{\nu_i} = b_{\nu_i}$ for $1 \leqq i \leqq k$. After establishing Ramsey's theorem without the use of Zermelo's axiom, the authors prove the following generalization: Let Δ be an arbitrary distribution of all sets of n positive integers into classes (not necessarily finite in number). Then there is an infinite set N^* of positive integers, and a canonical distribution $\Delta^{(k)}_{\nu_1, \ldots, \nu_k}$ which coincides with Δ as far as subsets of N^* are concerned. Let A, B, C, D denote sets of n positive integers. As a corollary of their theorem, the authors show that a distribution Δ is canonical if and only if, whenever A and B are in the same class of Δ and there exists a mapping of A onto C, B onto D which preserves order on $A \cup B$, then C and D are in the same class of Δ.

N. J. Fine (Philadelphia, Pa.).

Kneser, Hellmuth. Eine direkte Ableitung des Zornschen Lemmas aus dem Auswahlaxiom. Math. Z. 53, 110–113 (1950).

A direct (not using the well-ordering theorem) proof of the following: If, in a partly ordered set, each well-ordered subset has an upper bound, then there exists in the set a maximal element [see Hausdorff, Grundzüge der Mengenlehre, Veit, Leipzig, 1914, p. 140]. The proof is based on a lemma: Let f be a function on the partly ordered set X to itself. Suppose also that each well-ordered subset of X has an upper bound and that, for each x in X, $x \leqq f(x)$. Then there is an element x_0 such that $f(x_0) = x_0$. The theorem follows at once from the lemma using the axiom of choice. A slightly weaker form of the lemma has been stated by Bourbaki [Éléments de Mathématique, Théorie des ensembles, Actualités Sci. Ind., no. 846, Hermann, Paris, 1939, p. 37; these Rev. 3, 55].

A. D. Wallace.

Witt, E. On Zorn's theorem. Revista Mat. Hisp.-Amer. (4) 10, 82–85 (1950). (Spanish)

Expository lecture deriving Zorn's lemma directly from the axiom of choice.

Neumer, Walter. Über den Aufbau der Ordnungszahlen. Math. Z. 53, 59–69 (1950).

This article has a twofold purpose: to prove that there exist regular initial numbers whose indices are limit numbers, and to characterize the initial numbers in an exclusively ordinal manner [cf. Zermelo, Fund. Math. 16, 29–47 (1930); Kamke, Math. Z. 39, 112–125 (1934)]. To each of these ends, the author forms the (perhaps paradoxical) well-ordered "set" X of all ordinal numbers possessing a certain property P, calls Ξ the order type of X, and, operating with Ξ as an ordinal number, shows that no $\xi \geqq \Xi$ belongs to X, from which he concludes that X is not paradoxical. This argument, however, is incorrect, for it is begging the question to assume that Ξ is an ordinal number; e.g., if we take P to be the property of being an ordinal number, then X is the familiar "set" of all ordinal numbers. If we call Ξ its order type, then again no $\xi \geqq \Xi$ is in X. But from this we cannot conclude that X is not paradoxical. For it also follows, from the definition of X, that every $\xi \geqq \Xi$ is in X.

F. Bagemihl (Rochester, N. Y.).

Kozlova, Z. I. On coverings of certain A-sets. Izvestiya Akad. Nauk SSSR. Ser. Mat. 14, 421–442 (1950). (Russian)

Let I be the space of irrational numbers with its usual topology; let I_{xy} be the Cartesian product of two replicas I_x and I_y of I. For $x_0 \varepsilon I$, let P_{x_0} be the set of points in I_{xy} with fixed x-coordinate x_0. Let $\mathcal{E}$ be an A-set in I_{xy}. If $P_x \cap \mathcal{E}$ enjoys a certain set-theoretic or topological property K for all $x \varepsilon I$, one may ask if there exists a B-set H containing $\mathcal{E}$ such that $P_x \cap H$ has the property K for all $x \varepsilon I$. For various simple properties K (to consist of not more than one point, to be countable, etc.) affirmative answers have been obtained by other writers [see Glivenko, Rec. Math. [Mat. Sbornik] (1) 36, 138–142 (1929); Luzin, Leçons sur les ensembles analytiques . . . , Gauthier-Villars, Paris, 1930, p. 247]. In the paper under review, a rather complicated condition K is considered. A set E is said to be of absolutely the first class, if for every compact set B, $E \cap B$ contains a point of local compactness. Let E be a set in I_{xy} of absolutely first class. Write $E^{[0]} = E$. If α is an ordinal number $< \Omega$ and $\alpha - 1$ exists (Ω is the smallest uncountable ordinal), let $E^{[\alpha]}$ be the set obtained from $E^{[\alpha-1]}$ by removing a certain family of compact sets (the process used is not clearly defined). If $\alpha < \Omega$ and $\alpha - 1$ does not exist, then $E^{[\alpha]}$ is defined as $\bigcap_{\alpha' < \alpha} E^{[\alpha']}$. The least $\beta < \Omega$ such that $E^{[\beta]} = 0$ is called the subclass of E. Let $\mathcal{E}$ be a subset of I_{xy} such that every $P_x \cap \mathcal{E}$ is of absolutely first class. The least ordinal number α such that the subclass of $P_x \cap \mathcal{E} < \alpha$ for all $x \varepsilon I$ is called the order of $\mathcal{E}$. Suppose that $\mathcal{E}$ is an A-set in I_{xy} such that every $P_x \cap \mathcal{E}$ is of absolutely first class and such that the order of $\mathcal{E}$ is $\alpha < \Omega$. Then there exists a B-set H such that $H \supset \mathcal{E}$, every set $P_x \cap H$ is of absolutely first class, and the order of H is $\leqq \alpha$. A second result of the same general type is also proved.

E. Hewitt (Seattle, Wash.).

Nozaki, Yasuo. On generalized transfinite diameter. Kōdai Math. Sem. Rep. 1950, 3–10 (1950).

Let M be a bounded and closed set in space. In generalizing certain set functions introduced by Fekete [Math. Z. 17, 228–249 (1923)] and by Pólya and the reviewer [J. Reine Angew. Math. 165, 4–49 (1931)], the author defines the quantities $R(M)$ and $D(M)$ depending on M and on a function $\phi(r)$, $r > 0$. In the first place the following conditions are made: $\phi(+0) = \infty$; $\phi(+\infty) = 0$; $\phi(r)$ is decreasing and continuous. Then for each n we define $R_n = R_n(M)$ and $D_n = D_n(M)$ as follows:

$$\phi(R_n) = \max_{p_\nu} \min_{p \subset M} n^{-1} \sum_{\nu=1}^{n} \phi(r_{pp_\nu}),$$

where p_ν are arbitrary points in space. Further,

$$\phi(D_n) = \min_{p_\nu \subset M} \left[\binom{n}{2}\right]^{-1} \sum_{\mu < \nu} \phi(r_{p_\mu p_\nu}).$$

Following the argument of the authors cited above the existence of $\lim R_n = R$ and $\lim D_n = D$ can be shown as $n \to \infty$. Now it is proved that $R = D$ holds provided $\phi(r)$ is subject to the following additional conditions: $\phi(r)$ is convex; for each $c > 0$ one has $\lim \phi(cr)/\phi(r) = k > 0$ as $r \to 0$ and $\lim \phi(r+c)/\phi(r) = l > 0$ as $r \to \infty$.

G. Szegö.

Eggleston, H. G. The Besicovitch dimension of Cartesian product sets. Proc. Cambridge Philos. Soc. 46, 383–386 (1950).

The Besicovitch dimension of a set A in a Euclidean space is that number β such that the Hausdorff α-dimensional measure of A is zero if $\alpha > \beta$ and is infinity if $\alpha < \beta$.

The author proves that this definition of dimension satisfies the inequality: $\dim (A\times B)\geqq \dim A+\dim B$, where $A\times B$ is the Cartesian product of the sets A and B.

L. H. Loomis (Cambridge, Mass.).

Freilich, Gerald. On the measure of Cartesian product sets. Trans. Amer. Math. Soc. **69**, 232–275 (1950).

This paper is concerned with the following problem. Let A and B be orthogonal subspaces of Euclidean n-dimensional space of dimensions α and β, respectively, and with $\alpha+\beta=n$. Let M_n^k denote a measure of dimension k defined over subsets of Euclidean n-space E^n. If $A_1\subset A$, $B_1\subset B$, does $M_\alpha^m(A_1)\cdot M_\beta^k(B_1)=M_n^{m+k}(A_1\times B_1)$ (where $\times$ denotes the Cartesian product)? Only integral values of k and m are considered and the measures used of dimension k in E^n are Favard measure $\mathfrak{F}_n^k$, Hausdorff measure $\mathfrak{H}_n^k$, and Carathéodory measure C_n^k, defined as follows. Let R be an orthogonal transformation of E^n onto itself, G_n the compact topological group of all such transformations, and ϕ_n the Haar measure in G_n for which $\phi_n(G_n)=1$. Let $\mathfrak{L}_k(R,X)$ denote the k-dimensional Lebesgue measure of all those points whose first k coordinates are the same as those of $R(x)$ for some $x\epsilon X$ and whose last $n-k$ coordinates are zero, where X is a subset of E^n. Write

$$g_1(X)=(k!)^{-1}(\Gamma(\tfrac{1}{2}))^{k-1}\Gamma(\tfrac{1}{2}(k+1))(\operatorname{diam} X)^k,$$

$$g_2(X)=\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2}(n+1))\{\Gamma(\tfrac{1}{2}(k+1))\Gamma(\tfrac{1}{2}(n-k+1))\}^{-1} \times\int_{G_n}\mathfrak{L}_k(R,X)d\phi_n(R),$$

$$g_3(X)=\sup_{R\epsilon G_n}\mathfrak{L}_k(R,X).$$

Let $Y_1^{(X)}$, $Y_2^{(X)}$, $Y_3^{(X)}$ be three countable families of subsets of E^n such that the point set sum of the members of each family contains X, each member is of diameter less than δ, and the members of Y_2 are analytic and those of Y_3 are convex open. Let $Z_i(X,\delta)$ be the class of all such $Y_i(X)$, $i=1, 2, 3$. Write

$$\mathfrak{G}_i(X)=\lim_{\delta\to 0}\{\text{lower bound}_{Y_i(X)\epsilon Z_i(X,\delta)}\{\sum_{Y_i(X)} g_i(Y)\}\}.$$

Then $\mathfrak{F}_n^k(X)=\mathfrak{G}_1(X)$, $\mathfrak{H}_n^k(X)=\mathfrak{G}_2(X)$, and $C_n^k(X)=\mathfrak{G}_3(X)$. The author shows that the problem has an affirmative answer for $\mathfrak{F}_n^k$ when (i) $k=1$, $1\leqq m\leqq n-1$ or (ii) $k=\beta$, $1\leqq m\leqq\alpha=\beta-n$ or (iii) A_1, B_1 satisfy certain Lipschitz or rectifiability conditions. It is known that the answer is negative for $\mathfrak{H}_n^k$ when $n=3$, $\alpha=2$, $\beta=1$, and B_1 is a linear segment [Besicovitch and Moran, J. London Math. Soc. **20**, 110–120 (1945); these Rev. **8**, 18]. The author gives a detailed account of a set which illustrates this fact. The author also gives a formula relating $\mathfrak{F}_n^k(A)$ to an integral over E^m, $k\leqq m\leqq n$, with respect to $\mathfrak{F}_m^k$ measure.

H. G. Eggleston (Swansea).

Moore, Edward F. Density ratios and $(\phi, 1)$ rectifiability in n-space. Trans. Amer. Math. Soc. **69**, 324–334 (1950).

A set A of Euclidean n-dimensional space E_n is rectifiable if there exists a function $f(x)$ whose values include A, whose domain is a bounded subset of the real numbers, and for which $|f(x)-f(y)|\leqq|x-y|$. The author shows that, if ϕ is a measure such that $\phi(E_n)<\infty$ and A is a set such that at ϕ almost all points of A the ratio of the upper to the lower spherical density of A is less than 1.01, then A is $(\phi, 1)$ rectifiable, i.e., rectifiable if we ignore a subset of A whose ϕ-measure is arbitrarily small. The method is to construct a compact connected set of finite Hausdorff one-dimensional measure A_1 containing A. It follows from a known theorem [Eilenberg and Harrold, Amer. J. Math. **65**, 137–146 (1943); these Rev. **4**, 172] that A is rectifiable. Other related results in terms of different densities are also given.

H. G. Eggleston (Swansea).

Leipnik, Roy B. Note on preservation of measurability. Proc. Amer. Math. Soc. **1**, 694 (1950).

The author proves a sufficient condition for a subset of a set Y to be measurable with respect to an outer measure of the form mf, where m is an outer measure on a set X and f maps subsets of Y into subsets of X.

H. M. Schaerf.

Rindung, Ole. A proof of the possibility of dividing the real numbers into more than countably many disjoint and congruent sets with positive outer Lebesgue measure. Mat. Tidsskr. B. **1950**, 16–17 (1950). (Danish)

For any set E of reals, let E_t be the translation of E by t and let (E) be the set of linear combinations of E with non-zero rational coefficients. Express a Hamel basis B for the reals as the sum of a sequence of disjoint noncountable subsets, and let S and T denote the sum to n terms and the remainder. Then the set R of all reals has the decomposition $R=\sum_{t\epsilon(T)}(S)_t$ with the properties stated if n is large.

L. C. Young (Madison, Wis.).

Denjoy, Arnaud. Les applications du théorème général de Vitali. C. R. Acad. Sci. Paris **231**, 737–740 (1950).

This note is a sequel to two previous notes [same vol., 560–562, 600–601 (1950); these Rev. **12**, 246]. The sets ω and γ are identified, i.e., $\gamma=\gamma(\omega)=\omega=\omega(\gamma)$. On the fundamental set U, there is a nonnegative measure function $\varphi(E)$, completely additive and subtractive on measurable subsets of U. By the greatest lower and least upper bound process there are derived outer and inner measures $\varphi_e(E)$ and $\varphi_i(E)$. The Vitali theorem is then based on a family G of sets γ called regular relative to φ if (1) $\varphi(\gamma)>0$ for each γ of G; (2) for any γ, the set $\varphi(\gamma)$ not in γ indefinitely covered by sets γ' of G intersecting γ is of zero φ-measure; (3) there exist two numbers $1<a<b$ independent of γ such that if $\Omega(\gamma)$ is the sum of the sets γ' which intersect γ and for which $\varphi(\gamma')<a\varphi(\gamma)$, then $\varphi_e(\Omega(\gamma))<b\varphi(\gamma)$; (4) $\varphi_e(\sum\gamma)<\infty$. For the set $\Delta=\Delta(G)$ which is the set of all points of U indefinitely covered by the sets γ of G, the Vitali theorem is stated in the form: For every $\epsilon>0$, there exists a sequence of disjoint sets γ_n, such that if $\Gamma=\sum\gamma_n$, then (1) $\varphi(\Delta-\Delta\cdot\Gamma)=0$; (2) $\Delta(G)$ is φ-measurable and $\varphi(\Delta)=\varphi(\Delta\cdot\Gamma)$; (3) $\varphi(\Gamma)<\varphi(\Delta)+\epsilon$. For a set function $\psi(E)$ defined in the field $\Delta(G)$ (i.e., $\psi(E)$ has a definite value for every φ-measurable subset of $\Delta(G)$) it is possible to define derivatives at points M by considering the quotient $\psi(\Delta\cdot\gamma)/\varphi(\gamma)$, γ containing M. The Vitali theorem is then used to derive theorems paralleling the metric density theorem, the fact that a set of points where a monotonic bounded function has an infinite derivative is of zero measure, and the fact that absolutely continuous functions have a finite derivative almost everywhere.

T. H. Hildebrandt (Ann Arbor, Mich.).

Jessen, Børge. A remark on strong differentiation in a space of infinitely many dimensions. Mat. Tidsskr. B. **1950**, 54–57 (1950).

A well-known theorem asserts that if F is the indefinite integral of an arbitrary bounded measurable function f defined in a Euclidean space R_n, then the strong derivative

F' of F equals $f(x)$ at almost all points x [cf. Saks, Theory of the Integral, 2d ed., Stechert, New York, 1937, p. 132]. The author shows that in this theorem R_n cannot be replaced by the torus space of infinitely many dimensions if $F'(x)$ is defined to be the limit of $F(I_x)/m(I_x)$, where I_x is an arbitrary interval containing x whose kth side is both parallel to the kth coordinate axis and has a length converging to 0 for every k (but not uniformly in k).

H. M. Schaerf (St. Louis, Mo.).

Hausner, Melvin. A certain property of continuous functions. Pi Mu Epsilon J. 1, 15–17 (1949).

The author considers the set C of continuous functions $f(x)$ on $0 \leqq x \leqq 1$ which vanish at 0 and 1, and he calls $h>0$ a sub-period of an element $f(x)$ of C if the equation $f(x)=f(x+h)$ has a solution on $0 \leqq x \leqq 1-h$. He shows that a necessary and sufficient condition that h be a sub-period of every element of C is that it be the reciprocal of a positive integer.

R. H. Cameron (Minneapolis, Minn.).

Haupt, Otto. Wronskische Determinante und lineare Abhängigkeit. Math. Z. 53, 122–130 (1950).

A generalization of the classical result on the relation between linear dependence and the vanishing of the Wronskian. The generalization consists in treating functions on a locally connected space to a topological field. A "differentiation" satisfying suitable conditions is postulated.

A. D. Wallace (New Orleans, La.).

— See the note on p. 1002, Errata.

Conti, Roberto. Sulla derivata dell'integrale. Boll. Un. Mat. Ital. (3) 5, 128–133 (1950).

If $f(t)$ is bounded and measurable on $a \leqq t \leqq b$ and $F(x)=\int_a^x f(t)dt$, each of the following three conditions are sufficient in order that $F_+'(x)=0$: (A) The function $(t-x)^{-1}F(t)$ is absolutely integrable on $x \leqq t \leqq b$;

(B) $$\lim_{h\to+0}\int_{x+h}^b (t-x)^{-1}F(t)\,dt$$

exists; (C) $F(t)$ is approximately infinitesimal on the right at $t=x$, i.e., the set of points for which $t \geqq x$ and $|f(t)|<\epsilon$ has unit density for every $\epsilon>0$. The note points out that if $F(t)$ satisfies (A) at $t=x$, it also satisfies (B) and (C), but there exist functions satisfying (B) and (C) and not (A). Also the conditions (B) and (C) are independent.

T. H. Hildebrandt (Ann Arbor, Mich.).

Conti, Roberto. Sul secondo teorema della media per gli integrali doppi. Rend. Sem. Mat. Univ. Padova 19, 294–302 (1950).

Given an interval R of the (x, y)-plane with two sides on the axes (say), the author defines a certain class Δ of subdomains D of R and states that Δ is compact, closed, and connected. This statement appears to be incorrect and the reviewer presumes that the definition of Δ should read: $D\epsilon\Delta$ means that D is a subdomain of R whose boundary consists of two segments on the two axes together with a simple arc whose intersection with every parallel to the axes is connected. This being so, let L be the class of real functions $f(x, y)$ Lebesgue integrable on R, and let P be the subclass of L consisting of bounded nonnegative functions $p(x, y)$ which are monotone decreasing in each of the variables x and y; for any such $p(x, y)\epsilon P$, let $\bar{p}$ be an arbitrarily chosen nonnegative constant not exceeding the essential maximum of $p(x, y)$. The author's main theorem is that, given $f\epsilon L$, $p\epsilon P$, $\bar{p}$ as explained, the formula $\int_R pfd(x, y)=\bar{p}\int_D fd(x, y)$ holds for some $D\epsilon\Delta$. This supplements two classical results: that of Lebesgue (in which P is enlarged by removing the condition of monotony and Δ consists of the measurable subsets of R); and that of W. H. Young (in which P is made smaller by the further condition that the double increment of $p(x, y)$ in (x, y) be nonpositive and Δ consists of the subintervals of R with two sides on the axes).

L. C. Young (Madison, Wis.).

Theory of Functions of Complex Variables

Lehner, Joseph. Note on power series with integral coefficients. J. London Math. Soc. 25, 279–282 (1950).

This note contains a generalization of a result due to Graetzer [same J. 22, 90–92 (1947); these Rev. 9, 179] who proved that any complex number of absolute value less than unity is a zero of a power series with rational integral coefficients and radius of convergence unity. The generalization states that, if E is a countable set of complex numbers lying within the unit circle, then there is a power series, with rational integral coefficients and radius of convergence unity, which vanishes on the set E and on the set $\bar{E}$ of complex conjugates, but nowhere else in $|z|<1$. The author observes that no transcendental number of modulus greater than unity is the zero of a function represented by a power series (about $z=0$) with integral coefficients.

M. S. Robertson (New Brunswick, N. J.).

Sz.-Nagy, Gyula. Totalreelle rationale Funktionen. Acta Sci. Math. Szeged 12, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 1–10 (1950).

By a totally real rational function, the author means the quotient $F(z)=f(z)/g(z)$ of two nth degree, real polynomials $f(z)=\sum_0^n a_k z^{n-k}$ and $g(z)=\sum_0^n b_k z^{n-k}$ which are relatively prime with $|a_0|+|b_0|\neq 0$ and which are real only on the real axis. His first theorem gives five conditions, each of which is necessary and sufficient for $F(z)$ to be totally real. One of these, for example, is that the zeros of $F(z)-A$ separate those of $F(z)-B$, $A\neq B$, and vice-versa. A second theorem establishes that certain combinations of totally real rational functions are functions of the same kind. Two examples are $F_0(z)=[AF(z)+B]/[Cf(z)+D]$ with $AD\neq BC$; $\Phi(z)=[\sum_0^n a_k P^{(k)}(z)]/[\sum_0^n b_k P^{(k)}(z)]$, where $P^{(k)}(z)$ is the kth derivative of a polynomial $P(z)$ having only real zeros. Concerning the value distributions of totally real rational functions, the author proves a result analogous to one on polynomials due to Fekete [Jber. Deutsch. Math. Verein. 34, 220–233 (1925)]; namely, that if $F(a)=F(b)$ for the two real numbers a and b $(a<b)$ then $F(z)$ assumes every value in the lens-shaped region comprising all points z from which segment ab subtends an angle not less than π/n. Further theorems concern some mapping properties of a totally real $F(z)$ including the location of the fixed points of the transformation $w=F(z)$. The proofs of all the theorems are quite elementary.

M. Marden (Milwaukee, Wis.).

Noble, M. E. A theorem on the zeros of integral functions. J. London Math. Soc. 25, 282–284 (1950).

The author proves that there is a constant $B \leqq \frac{1}{2}(3+\sqrt{5})$ such that if $f(z)$ is an entire function of order $\rho<1$ such that all of its zeros lie in the strip $|x|<A$, then all of the zeros of f' lie in the strip $|x| \leqq AB$. [The reviewer would

like to point out that a much shorter proof using the classic theorems of Hurwitz and Lukacs shows that $B=1$.]

R. C. Buck (Madison, Wis.).

Lauritzen, Svend. On entire transcendental functions which approach a definite limit along every ray from the origin. Mat. Tidsskr. B. **1950**, 42–48 (1950). (Danish)

The author gives new proofs for two results of A. Roth [Comment. Math. Helv. 11, 77–125 (1938)] concerning the functions described in the title. In any angle in which the function is bounded, its limit along rays is constant. Given an arbitrary set of disjoint open intervals on the unit circle, and to each interval a number, there is an entire function which approaches a limit along every ray, approaches the given number in each given interval, and is bounded in each subinterval of each given interval but not in any larger interval.

R. P. Boas, Jr. (Evanston, Ill.).

Revuz, André. Sur le théorème de Denjoy-Carleman-Ahlfors. C. R. Acad. Sci. Paris **231**, 817–819 (1950).

The author shows how the proof by which Ahlfors established Denjoy's conjecture that an entire function of order ρ has at most 2ρ asymptotic values [Acta Soc. Sci. Fennicae. N.S. A. 1, no. 9 (1930)] can be modified to establish Denjoy's further conjecture that $f(z)$ has at most 2ρ asymptotic polynomials.

R. P. Boas, Jr. (Evanston, Ill.).

***Schaeffer, A. C., and Spencer, D. C. Coefficient Regions for Schlicht Functions. With a Chapter on the Region of the Derivative of a Schlicht Function by Arthur Grad.** American Mathematical Society Colloquium Publications, Vol. 35. American Mathematical Society, New York, N. Y., 1950. xv+311 pp. $6.00.

Let S denote the class of functions

$$f(z)=z+a_2z^2+\cdots+a_nz^n+\cdots$$

which are regular and schlicht in $|z|<1$. If $a_k=\alpha_k+i\beta_k$ the point $(a_2, \cdots, a_n)$ is said to belong to the region V_n in $(2n-2)$-dimensional real Euclidean space with coordinates $(\alpha_2, \beta_2, \cdots, \alpha_n, \beta_n)$ if there is a function

$$f(z)=z+b_2z^2+\cdots+b_nz^n+\cdots$$

of class S for which $b_k=a_k$ for $k\leqq n$. The coefficient problem for schlicht functions, proposed by Bieberbach about 1916, is to find for each $n\geqq 2$ the precise region V_n. The present work is largely confined to this problem, although it is only one of a wide class of problems of the family S to which the powerful variational methods developed here may be applied.

A brief history of the problem is given in the first chapter and several of the elementary properties of V_n are obtained by simple methods. For example, V_n is a bounded, closed domain, topologically equivalent to the closed $(2n-2)$-sphere. A goal to be obtained in later chapters is to show that the boundary of V_n is composed of a finite number of pieces such that the coordinates of a point $(a_2, \cdots, a_n)$ on any one of these pieces will be functions of a finite number $(\leqq 2n-3)$ of parameters. To each function f of S a wide class of neighboring functions $f^*(z, \epsilon)$ of class S, $f^*=f+O(\epsilon)$, is developed in the second chapter. The method is a modification of one published before [Duke Math. J. **14**, 949–966 (1947); these Rev. **9**, 341] and includes as a special case formulae obtained earlier by Schiffer [Amer. J. Math. **65**, 341–360 (1943); these Rev. **4**, 215].

The variational formulae are then used to obtain information concerning the functions f corresponding to boundary points $(a_2, a_3, \cdots, a_n)$ of V_n. Such a function satisfies a differential equation of the form (A) $(z/w)(dw/dz)^2P(w)=Q(z)$, where $P(w)=\sum_{\nu=1}^{n-1}A_\nu/w^\nu$, $Q(z)=\sum_{-(n-1)}^{n-1}B_\nu/Z^\nu$, and $Q(z)\geqq 0$ on $|z|=1$ with at least one zero on $|z|=1$ of even order. Chapter three continues with a study of the solutions $w=z+a_2z^2+\cdots$ of (A) regular in $|z|<1$. Because (A) is separable, it is perhaps natural to study the loci Γ_w, defined by $\Re\int(P(w))^{\frac{1}{2}}w^{-1}dw=c$, c a constant, which turn out to be the boundary in the w-plane corresponding to the map of $|z|=1$. A detailed account of the nature of these loci is given, and in the following chapter a study of the behavior in the large for this Γ structure is pursued. Introducing in chapter five a metric defined by $|d\zeta|^2=|P(w)/w^2|\cdot|dw|^2$, the authors show that any two points $w_1\neq 0$ and $w_2\neq 0$ are connected by a unique geodesic.

In looking for the solutions of (A) the normalization $f'(0)=1$ leads to $A_k=B_k$ and $A_\nu=0$ for $\nu>k$, where k is the largest integer for which $B_k\neq 0$. With these properties (A) is called a $\mathfrak{D}_n$-function. In chapter six it is then shown that if $w=f(z)$ is a $\mathfrak{D}_n$-function, it must be schlicht in $|z|<1$, mapping $|z|<1$ onto the w-plane minus a subcontinuum of Γ_w containing $w=\infty$. Moreover, $f(z)$ is regular on $|z|=1$ except for a finite number of points. Using the length-area principle of Teichmüller the authors complete in chapter seven an earlier theorem with its converse. There is a one-to-one correspondence between points on the boundary of V_n and $\mathfrak{D}_n$-functions. Every $\mathfrak{D}_n$-function belongs to some boundary point of V_n and to any given boundary point of V_n there belongs precisely one $\mathfrak{D}_n$-equation. However, there is no one-to-one correspondence between $\mathfrak{D}_n$-functions and $\mathfrak{D}_n$-equations. The $\mathfrak{D}_n$-equation is determined by a pair of vectors $\mathfrak{A}=(A_1, \cdots, A_{n-1})$, $\mathfrak{B}=(B_1, \cdots, B_{n-1})$. Given $\mathfrak{A}$, does there exist a $\mathfrak{B}$ so that (A) is a $\mathfrak{D}_n$-equation with a $\mathfrak{D}_n$-function as a solution and when is $\mathfrak{B}$ unique? This question and special properties of the equation (A) are explored in chapter eight.

In chapter nine Löwner's differential equation is obtained by the new methods. Its solution $v(z, u)=u\{z+b_2(u)z^2+\cdots\}$, $0<u\leqq 1$, gives rise to a one-parameter family of functions $v(z, u)$ which defines a curve from a given $\mathfrak{D}_n$-function $f(z)$ to the function z. Thus the point $(b_2(u), \cdots, b_n(u))$ defines a curve in V_n from the point $(a_2, \cdots, a_n)$ associated with $f(z)$ to the origin. For a given set of constants $\lambda_2, \cdots, \lambda_n$ not all zero, let the maximum of $F=2\Re\{\lambda_2b_2+\cdots+\lambda_nb_n\}$ in V_n be M. The region V_n lies on one side of the supporting hyperplane $F=M$. The subset K_n of V_n in which a supporting hyperplane can touch V_n is called the supporting surface. For $n>2$, K_n is a proper subset of the boundary of V_n. It is shown in chapter ten that K_n is connected and that any function $w=f(z)$ belonging to a point of K_n maps $|z|<1$ onto the plane minus a single analytic slit.

Chapter eleven is devoted to a portion of the boundary of V_n corresponding to single analytic slits. Let M_n denote the set of boundary points of V_n associated with functions $f(z)$ which satisfy only one $\mathfrak{D}_n$-equation and such that in this equation $Q(z)$ vanishes on $|z|=1$ at only one point where it has a zero of order two. The set M_n is not empty, is open and its closure contains K_n. There is a one-to-one bicontinuous correspondence between points of M_n and this special class of Q-functions. A variational formula is developed for nearby functions $f^*(z)$ associated with a point a^* of M_n near the point a of M_n associated with $f(z)$. The image of $|z|=1$ by a function $w=f(z)$ belonging to a boundary point of V_n is always a tree. Chapter twelve gives a method of describing the tree in terms of a finite

number of parameters. A formula for the number of topologically different trees having a fixed number of tips is given. However, the parametrization of trees here depends also on the lengths of its branches in terms of a metric.

In chapter thirteen the precise parametric equations of the boundary of V_3 are obtained in terms of elementary functions. This boundary is composed of essentially two hypersurfaces of dimension three and their two-dimensional intersection. The appendix contains tables for the computation of the boundary of V_3. Although the region is four-dimensional a rotational property permits a complete description from certain three-dimensional cross sections. It has long been conjectured that $|a_n|\leqq n$ for the family S of schlicht functions with equality only for the Koebe functions $f(z)=z(1-e^{i\theta}z)^{-2}$, θ real. The problem is equivalent to that of finding the maximum distance of the boundary of V_n from the hyperplane $a_n=0$. In chapter fourteen a method is derived for proving or disproving the conjecture in the case $n=4$. The method involves a great amount of computation and the results were not at this time ready for announcement. The program outlined consists of three parts: (a) proof that the points of the boundary of V_4 corresponding to the Koebe functions have the local maximum property with respect to the neighboring points; (b) proof that over a large part of the boundary $|a_4|\leqq 4-\delta$; (c) calculation of points on the remaining portion of the boundary for which $|a_4|>4-\delta$ by integrating a system of differential equations.

The concluding chapter was written by A. Grad who obtained the region of values of the derivative $f'(z)$ of a schlicht function. The differential equation for functions $f(z)$ whose derivative at a point z lies on the boundary of the region of values involves essentially one real parameter. Its solutions may be expressed in terms of elementary functions. It is convenient to consider the region of values of $f'(z)$ in the plane of $\log f'(z)$. This region lies within a rectangle and Grad's solution determines the exact region within this rectangle. *M. S. Robertson.*

Dvoretzky, Aryeh. Bounds for the coefficients of univalent functions. Proc. Amer. Math. Soc. 1, 629–635 (1950).

Let $w=f(z)=z+a_2z^2+\cdots+a_nz^n+\cdots$ be regular and schlicht in $|z|<1$, W the image of $|z|<1$ by $w=f(z)$, and $A(R)$ the radius of the largest circle on $|w|=R$ the whole interior of which is contained in W. If $\limsup_{R\to\infty} A(R)/R\leqq L$, it is shown that there exists a finite positive K such that $\limsup_{n\to\infty} |a_n|/n\leqq KL$. Moreover, if $A(R)=O(R^\gamma)$, it is shown that: $a_n=O(n^{\gamma/(2-\gamma)})$, $0<\gamma\leqq 1$; $a_n=O(\log n)$, $\gamma=0$; $a_n=O(1)$, $-2\leqq\gamma<0$; $a_n=O(n^{-\frac{1}{2}+2/(2-\gamma)})$, $\gamma<-2$.
D. C. Spencer (Princeton, N. J.).

Nagura, Shohei. Faber's polynomials. II. Kōdai Math. Sem. Rep. 1950, 15–16 (1950).

[For part I see the same Rep. 1949, no. 5–6, 5–6 (1949); these Rev. 11, 718.] Let $f(z)=z+a_2z^2+a_3z^3+\cdots$ be schlicht in the unit circle. The author derives Löwner's inequality $|a_3|\leqq 3$ by means of a procedure which does not differ appreciably from Löwner's original proof (which is quoted).
Z. Nehari (St. Louis, Mo.).

Grötzsch, Herbert. Zur Geometrie der konformen Abbildung. Hallische Monographien no. 16, pp. 5–11. Max Niemeyer Verlag, Halle (Saale), 1950. 4.80 RM.

Let E_3 denote the plane punctured at -1, 1, and ∞, and let S be a family of curves that covers E_3 and satisfies certain restrictions of a very general nature. Let B be an arbitrary schlicht region, of finite connectivity $n+3$, and having boundary elements $R_1,\cdots,R_n$, -1, 1, and ∞. The author proves that there exists a unique one-to-one conformal mapping of B which maps each of the boundary elements -1, 1, and ∞ into itself, and each of the boundary elements R_k into a slit on a curve S_k of the family S. If R_k consists of a single point, the same is true of its image. The proof is similar to that used in an earlier paper by the author, where the case of the twice-punctured plane is treated [Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Phys. Kl. 87, 145–158, 159–167 (1935)]. *G. Piranian.*

Warschawski, S. E. On the degree of variation in conformal mapping of variable regions. Trans. Amer. Math. Soc. 69, 335–356 (1950).

Soient R_1, R_2 deux domaines simplement connexes "voisins" contenant l'origine $w=0$. Soient $w=f_1(z)$, $w=f_2(z)$ les fonctions qui représentent conformément le cercle $|z|<1$ sur R_1, R_2 respectivement, de telle sorte que $f_1(0)=f_2(0)=0$, $f_1'(0)>0$, $f_2'(0)>0$. Le problème de majoration de la quantité $|f_1(z)-f_2(z)|$ en fonction de la distance des frontières B_1, B_2 de R_1, R_2 avait été déjà étudié par divers auteurs [Bieberbach, S.-B. Preuss. Akad. Wiss. 1924, 181–188; Marchenko, C. R. (Doklady) Acad. Sci. URSS (N.S.) 6 (1935 I), 289–290; Markouchevitch, Rec. Math. [Mat. Sbornik] N.S. 1(43), 863–886 (1936); J. Ferrand, Bull. Soc. Math. France 70, 143–174 (1942); C. R. Acad. Sci. Paris 221, 132–134 (1945); ces Rev. 6, 207; 7, 201] et résolu dans de cas où R_1 est un cercle, un domaine limité par une courbe de Jordan assez régulière ou un polygone. Dans tous ces cas on a $|f_1(z)-f_2(z)|<A\epsilon\log\epsilon^{-1}+B\eta(\epsilon)$ en désignant par ϵ la distance de B_1 à B_2 et par $\eta(\delta)$ la borne supérieure des diamètres des sous-domaines de R_2 séparés de l'origine par une transversale de diamètre au plus égal à δ; A et B sont des constantes positives. L'auteur étend ces résultats, sous une forme moins précise, au cas de domaines R_1, R_2 quelconques, grâce à une étude assez poussée des modules de structure $\eta_1(\delta)$, $\eta_2(\delta)$ de ces domaines, et des modules de continuité que l'on peut en déduire pour les transformations considérées; si l'on désigne par ϵ la distance "intérieure" des frontières, on a pour ϵ assez petit:

$$|f_1(z)-f_2(z)|\leqq[1+k\sigma^{-\frac{1}{2}}\epsilon^{\frac{1}{2}}\log(4/\epsilon)]$$
$$\times\{\eta_1[(8\pi A_1/\log(1/\epsilon))^{\frac{1}{2}}]+\eta_2[(8\pi A_2/\log(1/\epsilon))^{\frac{1}{2}}]\}$$

où σ désigne la distance de $w=0$ à B_1 et B_2, A_1, A_2 les aires de R_1, R_2 et k une constante absolue. L'auteur améliore cette inégalité (a) en supposant que $\eta_1(\delta)\eta_2(\delta)$ satisfont à $\eta_1(\delta)\leqq k\delta+\eta_1$, $\eta_2(\delta)\leqq k\delta+\eta_2$, (b) dans le cas où B_1 et B_2 sont des courbes de Jordan dont l'angle de la tangente a un module de continuité déterminé. Pour les transformations inverses, en supposant $R_1\subset R_2$, on a dans le cas général: $|f_1^{-1}(w)-f_2^{-1}(w)|\leqq L[\eta(\epsilon^{1/6})]^{\frac{1}{2}}$. *J. Lelong-Ferrand.*

Hirschman, I. I., Jr. On non-uniformly quasi-analytic functions. Amer. J. Math. 72, 863–867 (1950).

The author extends to nonuniformly quasianalytic functions some of his previous results on uniformly quasi-analytic functions. The following is a typical result: Let $f(x)$ satisfy $|f^{(n)}(x)|\leqq A[k(x)]^n n![\log(n+e)]^n$, $n\geqq 0$, where $k'(x)[k(x)]^{-2}=o(1)$ as $x\to+\infty$; if

$$f(x)=O\left(\exp\left[-\exp\exp\tfrac{1}{2}\pi(1+\epsilon)\int_0^x k(t)dt\right]\right),\quad x\to+\infty,$$

with $\epsilon>0$, then $f(x)\equiv 0$. The method of proof consists in reducing the nonuniform case to the uniform one ($k(x)\equiv$constant) by using conformal mapping. *S. Agmon.*

Leja, F. Une nouvelle démonstration d'un théorème sur les séries de fonctions analytiques. Actas Acad. Ci. Lima 13, 3–7 (1950).

Consider a series of the form $\sum_0^\infty f_n(x)y^n$, where x and y are complex variables and $f_1(x), f_2(x), \cdots$ are regular functions throughout a domain D in the x-plane. Let R_0 be the upper bound of all the numbers R such that $\sum f_n(x)R^n$ is convergent for all x in D and r_0 the upper bound of all numbers r such that $\sum f_n(x)r^n$ is uniformly convergent in some neighborhood of each x of D. Hartogs [Math. Ann. **62**, 1–88 (1906), p. 8] has shown that if $R_0>0$, then either $r_0=0$ or $r_0=R_0$. The author gives a new and simple proof of this result based on his theorem on the uniform boundedness of a sequence of polynomials [ibid. **108**, 517–524 (1933), p. 520]. He also gives an example in which $r_0=0$ but $R_0>0$.
A. C. Offord (London).

✱Fuks, B. A. Teoriya analitičeskih funkcil mnogih kompleksnyh peremennyh. [Theory of Analytic Functions of Several Complex Variables]. OGIZ, Moscow-Leningrad, 1948. 472 pp.

During the last thirty years, the theory of functions of several complex variables has been enriched by a number of new chapters which are of fundamental importance, and which seem to point in the direction of future developments in the field. It is the object of the present work to give a complete survey of the theory of functions of several complex variables, incorporating these more recent results with the now classical portions.

Chapter 1 gives the basic definitions and properties of analytic functions. In chapter 2, the author takes up the representation of analytic functions by means of various series developments. At first, power series are discussed, leading to the fundamental theorem of Hartogs and the various theorems of H. Cartan on circular regions, and, finally, developments in series of orthogonal functions, leading to the theory of Bergman's kernel function and its applications. Chapter 3 is devoted to a study of zero manifolds of analytic functions and includes discussions of the theorem of Weierstrass, properties of analytic surfaces and hypersurfaces, particularly from the point of view of differential geometry. Here also is discussed E. E. Levy's theory of analytic convexity. Chapter 4 considers singular points of analytic functions and the theorems of Hartogs and Osgood pertaining thereto.

Chapter 5 deals with regions of analyticity and meromorphicity. For this chapter, the results of such recent writers as Oka, Behnke, and Stein are included. Chapter 6 takes up the generalization due to Bergman and Weil of the Cauchy integral formula for domains with distinguished boundary surfaces. A discussion of various related questions, especially theorems of Oka and Bochner, is also given here. Chapter 7 considers Cousin's problems. In the final chapter, the author deals with the theory of pseudo-conformal transformations. After the older results of Poincaré and Osgood have been discussed, the metric of Carathéodory and that of Bergman are introduced. Applications of the Bergman kernel function to distortion theory, including Bergman's generalization of the lemma of Schwarz-Pick, are given here. This chapter also includes some interesting results of the author on pseudo-conformal transformations.

The bibliographical references to papers appearing in the decade prior to publication have, unfortunately, noticeable gaps, and a number of areas which have seen a rapid recent development are inadequately presented. However, it should be indicated that the exposition of the general theory is detailed, and the new material has been woven skillfully into the whole.
P. Davis (Cambridge, Mass.).

Fourier Series and Generalizations, Integral Transforms

Timan, A. F. A precise estimate of the remainder in the approximation of differentiable functions by Poisson integrals. Doklady Akad. Nauk SSSR (N.S.) **74**, 17–20 (1950). (Russian)

The author determines explicitly the best bound for the difference between a continuous function of period 2π and its Poisson integral $f(r, x)$, when $|f(x+h)-f(x)|\leqq M|h|$ or when $f(x)$ has a pth derivative with a prescribed bound. In the first case the result is

$$|f(x)-f(r,x)|\leqq 2M\pi^{-1}\left\{(1-r)\log\frac{1}{1-r}\right.$$
$$\left.+\int_0^{1-r}\left[\frac{1}{1-t}\log\frac{2-t}{t^t}+1\right]dt\right\}.$$

The asymptotic form of this bound has previously been obtained by Natanson [same Doklady (N.S.) **72**, 11–14 (1950); these Rev. **11**, 655]. As a corollary of the fact that the same estimate holds when $|f(x+h)+f(x-h)-2f(x)|\leqq 2M|h|$, the author gives the result that such a function which is of mean value zero over a period satisfies the (best possible) inequality $|f(x)|\leqq\frac{1}{2}\pi M$.
R. P. Boas, Jr.

Zamansky, Marc. Classes de saturation des procédés de sommation des séries de Fourier et applications aux séries trigonométriques. Ann. Sci. École Norm. Sup. (3) **67**, 161–198 (1950).

Some of the results established in the paper have been stated by the author previously without proof [see, in particular, C. R. Acad. Sci. Paris **229**, 695–696 (1949); **230**, 44–46 (1950); these Rev. **11**, 172, 348]. The following are some of the new results. (1) Let $f(x)$ be a continuous function of period 2π and $\tilde{f}(x)$ its conjugate. Then (a) a necessary and sufficient condition for $f^{(p)}\in\mathrm{Lip}\,1$ is $\Delta_{p+1}(f, x, t)/t^{p+1}=O(1)$, uniformly in x and t; (b) a necessary and sufficient condition for $\tilde{f}^{(p)}\in\mathrm{Lip}\,1$ is

$$\int_\epsilon^\infty t^{-(p+2)}\Delta_{p+2}(f, x, t)\,dt=O(1)$$

uniformly in x and $\epsilon>0$. The differences Δ here are defined by the conditions

$$\Delta_1(t)=f(x+t)-f(x-t),$$
$$\Delta_2(t)=f(x+t)-2f(x)+f(x-t),$$
$$\Delta_k(t)=\Delta_{k-2}(t)-2^{k-2}\Delta_{k-2}(\tfrac{1}{2}t),$$

$k=3, 4, \cdots$. (2) For the functions $f(x)$ whose best approximation by trigonometric polynomials of order n is $O(n^{-2})$, the existence of the second generalized derivative is equivalent to the existence of the ordinary second derivative.
A. Zygmund (Chicago, Ill.).

Matsumura, Yoshimi. On the summability of Fourier series. J. Osaka Inst. Sci. Tech. Part I. **1**, 91–95 (1949).

The author establishes a theorem of S. Izumi and T. Kawata [Tôhoku Math. J. **46**, 154–158 (1939); these Rev. **1**, 225] concerning the almost everywhere $(C, 1)$-summability of the sequence (*) $\{S_1^k(x), S_2^k(x), \cdots\}$ of partial

sums of a Fourier series. The almost everywhere (C, α)-summability of the sequence (*) for $x>0$ has been established by C. T. Loo [ibid. **48**, 215–220 (1941); these Rev. **10**, 247]. *P. Civin* (Eugene, Ore.).

Prasad, B. N., and Siddiqi, J. A. On the Nörlund summability of derived Fourier series. Proc. Nat. Inst. Sci. India **16**, 71–82 (1950).

The authors prove that any regular Nörlund method of summation (N, p_n) sums the derived series of a Fourier series to $f'(x)$ at every point where

$$\int_0^t u^{-1}|f(x+u)-f(x-u)-2uf'(x)|\,du=o(t)$$

provided $\sum_1^n|\Delta p_k|$, $\sum_1^n k|\Delta^2 p_k|$, and $\sum_1^n k^{-2}|P_k|$ are all three $O(n^{-1}P_n)$ as $n\to\infty$. The theorem is a generalization under less restrictive conditions on the p_k's of a result of Astrachan [Duke Math. J. **2**, 543–568 (1936)] who in turn extended results of the reviewer and Tamarkin [Trans. Amer. Math. Soc. **34**, 757–783 (1932)]. The latter, however, considered only the case of the derived series of functions of bounded variation for which the conditions could be given a simpler form. The authors' theorem also includes classical theorems on (C, r)-summability. The proof is based on a refinement of the methods of Astrachan, the reviewer, and Tamarkin. *E. Hille* (New Haven, Conn.).

Ščerbina, A. D. On a summation method of series conjugate to Fourier series. Mat. Sbornik N.S. **27(69)**, 157–170 (1950). (Russian)

Let $\bar{S}_n(x)=\bar{S}_n(x, f)$ be the nth partial sum of the series conjugate to the Fourier series of $f(x)$, and let

$$\bar{\sigma}(n, p)=\bar{\sigma}_n(n, p, f, x)=(\bar{S}_{n-p}+\cdots+\bar{S}_{n-1}+\bar{S}_n)/(p+1)$$

be the delayed $(C, 1)$ means of $\{\bar{S}_n\}$. (1) A necessary and sufficient condition that $\bar{\sigma}(n, p)$ be uniformly equiconvergent, for every continuous f, with

$$\bar{f}_n(x)=-\pi^{-1}\int_{1/n}^{\pi}[f(x+t)-f(x-t)]\,\tfrac{1}{2}\cot\tfrac{1}{2}t\,dt,$$

as $n\to\infty$, is that $\liminf_{n\to\infty} p/n>0$. (2) If the latter condition is satisfied, then $\sigma(n, p)\to\bar{f}(x)$ $(=\lim_{n\to\infty}\bar{f}_n(x))$ almost everywhere, for every integrable f. (3) A corresponding result is obtained for the series conjugate to the Fourier-Stieltjes series. *A. Zygmund* (Chicago, Ill.).

Schmetterer, Leopold. Taubersche Sätze und trigonometrische Reihen. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **158**, 37–59 (1950).

The author gives a number of loosely connected theorems in the fields indicated by the title. Some of his results are as follows. (I) If $f(x)$ is an even function whose Fourier coefficients a_n satisfy (1) $a_n=O(n^{-\delta})$, $0<\delta<1$, and (2) $\int_0^t|f(x)|dx=o(t/\log t^{-1})$ as $t\to 0$, while $g(x)$ is of bounded variation, then $f(x)g(x)$ satisfies (1) and (2) (which are the hypotheses of the Hardy-Littlewood convergence test). (II) Let s_n be V_α-summable (V for Valiron) if

$$\text{(3)}\qquad \lim_{n\to\infty}(2\pi)^{-\frac{1}{2}}n^{-\alpha}\sum_{m=-\infty}^{\infty}\exp\{-\tfrac{1}{2}m^2n^{-2\alpha}\}s_{m+n}$$

exists. Let $2\varphi(t)=f(x+t)+f(x-t)-2s$; then if the βth integral of $\varphi(t)$ is $o(t^{\beta+\epsilon})$ with $0<\beta\leqq 1$, $\epsilon>0$, $\beta/(\beta+\epsilon)<\alpha$, the Fourier series of $f(x)$ is V_α-summable to s at x. (III) If the expression under the limit sign in (3) is bounded, where s_n are the partial sums of $\sum a_n\cos nx$ at $x=0$, then $s_n=O(1)$ if $\liminf(s_n-s_m)>-b$ for $n-m=O(m^\alpha)$ [note misprints in the statement of the author's Satz 4]. (IV) If $\varphi_1(t)=\int_0^t\varphi(u)du=o(t^{1+\epsilon})$ and $\int_0^t|d(u\gamma\varphi(u))|=O(t)$ with $1+\epsilon\geqq\gamma>1$, the Fourier series of $f(x)$ converges at x; a larger γ will not do. (V) Call a series $\sum a_n$ L-summable if $\lim_{t\to 0}\sum a_n(nt)^{-1}\sin nt$ exists (a convergent series is not necessarily L-summable). If $s_k=\sum_{n=1}^k a_n$ and $S_n=\sum_{k=1}^n(s_k-s)$, then $S_n=o(n^{1-\alpha})$ and $a_n=O(n^{-\delta})$, $1-\alpha\leqq\delta<1$, together imply that $\sum a_n$ is L-summable to s. (VI) If $\sum_{k=1}^n ka_k=o(n)$ and $\sum_{k=n}^{2n}(a_k-|a_k|)=O(n^{1-\delta}\log n)$, $0<\delta<1$, then

$$\sum a_n(nt)^{-1}\sin nt=o(\log t^{-1}).$$

This enables the author to simplify the proof of a convergence criterion derived from $(R, \log n, 1)$ summability. *R. P. Boas, Jr.* (Evanston, Ill.).

***Morse, Marston, and Transue, William. The Fréchet variation and Pringsheim convergence of double Fourier series.** Contributions to Fourier Analysis, pp. 46–103. Annals of Mathematics Studies, no. 25. Princeton University Press, Princeton, N. J., 1950. $3.00.

This is a lucid exposition of the new results on the Pringsheim convergence of (rectangular) double Fourier series which the authors have been able to obtain with the aid of the technique that they have developed for the study of bilinear functionals over the space of Cartesian products $C\times C$ (or more general products). The connexion is provided by the fact that the partial sum of double Fourier series admits of a representation by the Dirichlet integral which, indeed, is one such functional. And the study of the limit-behaviour of this integral is usually made by postulating, directly or indirectly, on the Vitali variation either of a function or of an indefinite integral; here the concept of Fréchet variation is used instead, and this, being less restrictive, yields stronger results. Thus, corresponding to each one of several classical tests of convergence [cf. Gergen, Trans. Amer. Math. Soc. **35**, 29–63 (1933)], the authors establish a new and more delicate test. The inter-relations between the old and the new tests are thoroughly explored, and represented diagrammatically, and one or two gaps, existing in older literature, are filled. In settling the question of the relative strength of the various tests, the authors have had sometimes to construct new examples which mark a departure from the classical ones. *K. Chandrasekharan* (Bombay).

Wing, G. Milton. The mean convergence of orthogonal series. Amer. J. Math. **72**, 792–808 (1950).

M. Riesz [Math. Z. **27**, 218–244 (1927)] has investigated the convergence of the Fourier series of a function of the class L^p in the space L^p, $1\leqq p<\infty$. Similar questions were treated by other authors for various other orthogonal series. The present paper deals with Bessel series and with certain expansions in terms of orthogonal polynomials from this point of view. Let $\{\mu_n\}$ denote the positive roots of $J_\nu(x)$, $\nu\geqq-\frac{1}{2}$. If $f(x)\subset L^p$, $p>1$, $0\leqq x\leqq 1$, we have

$$\lim_{n\to\infty}\int_0^1|f(x)-S_n(x)|^p dx=0,$$

where S_n is the nth partial sum of the Fourier-Bessel series corresponding to the orthogonal functions $x^{\frac{1}{2}}J_\nu(\mu_n x)$. A similar theorem is proved for the Dini series, where the quantities μ_n are replaced by the roots of $xJ_\nu'(x)+HJ_\nu(x)$, H real. The results on orthogonal polynomial expansions generalize those of Pollard [Trans. Amer. Math. Soc. **62**, 387–403 (1947); **63**, 355–367 (1948); these Rev. **9**, 280, 426] dealing with Jacobi polynomials. *G. Szegö.*

Wilkins, J. Ernest, Jr. Neumann series of Bessel functions. II. Trans. Amer. Math. Soc. 69, 55–65 (1950).

[For part I see the same Trans. 64, 359–385 (1948); these Rev. 10, 249.] The author states 3 theorems on Neumann series of Bessel functions, which he has proved formerly. He now proposes to prove that these theorems are also valid if the class of functions to be developed is extended. In order to do this, he first proves that the two new classes of functions are identical. He then shows that the coefficients of the series all exist if the functions are in the second class. Thereupon he proves that the original theorems are valid for the two new classes of functions. *M. J. O. Strutt.*

Šreĭder, Yu. A. On the Fourier-Stieltjes coefficients of functions of bounded variation. Doklady Akad. Nauk SSSR (N.S.) 74, 663–664 (1950). (Russian)

The note states, without proofs, a number of results about the Fourier-Stieltjes coefficients $C_n[\varphi]=\int_0^{2\pi}e^{int}d\varphi(t)$ of functions $\varphi(t)$ of bounded variation over the interval $[0, 2\pi]$. Definition 1: A sequence of numbers $a_1, a_2, \cdots$ situated on the interval $[0, 1]$ is called Weyl-distributed, if it has a distribution function $\rho(t)$ $(=\lim_{k\to\infty} n(t, k)/k$, where $n(t, k)$ is the number of terms among $a_1, \cdots, a_k$ which fall into the interval $[0, t]$, $0<t<1$; a denumerable set of t's may be disregarded), and if $\rho(t)\neq t$ for some t. Definition 2: A point set E situated on the interval $[0, 2\pi]$ is called of type W, if there exists a sequence of integers $n_1<n_2<\cdots<n_k<\cdots$ such that for every $t\epsilon E$ the sequence of fractional parts $\{n_1t/2\pi\}$, $\{n_2t/2\pi\}, \cdots$ is Weyl-distributed. The main result of this note is as follows. A necessary and sufficient condition that $C_n[\varphi]\to 0$, as $n\to\infty$, is that the variation of $\varphi(t)$ over every set of type W be zero. *A. Zygmund.*

Barrucand, Pierre. Généralisation de la transformation de Stieltjes itérée: transformation d'ordre quelconque. C. R. Acad. Sci. Paris 231, 748–750 (1950).

Let $\omega_\alpha(s)=(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}(\pi \csc \pi s)x^{-s}ds$, where $0<c<1$, $0<\alpha$. The author establishes the formula

$$\omega_\alpha(x)=\frac{(2\pi)^\alpha\Gamma(\frac{1}{2}\alpha+i(\log x)/2\pi)\Gamma(\frac{1}{2}\alpha-i(\log x)/2\pi)}{4\pi^2x^{\frac{1}{2}}\Gamma(\alpha)}.$$

Using $s\Gamma(s)=\Gamma(s+1)$, there now results

$$\omega_{\alpha+2n}=\prod_{r=0}^{n-1}[\pi^2(\alpha+2r)^2+(\log x)^2]\times[\omega_\alpha(x)/\alpha(\alpha+1)\cdots(\alpha+2n-1)].$$

Other similar formulas are obtained. The function $\omega_n(x)$ is the nth iterate of the Stieltjes kernel. More generally, $\omega_\alpha(x)=\int_0^\infty\omega_{\alpha-1}(xt)\omega_1(t)\,dt$, where $\omega_1(t)=(1+t)^{-1}$. *I. I. Hirschman, Jr.* (St. Louis, Mo.).

González Domínguez, A. A contribution to the theory of Hille functions. Ciencia y Técnica 42, 283–329 (1941). (Spanish)

What the author calls Hille functions (better known as Gauss transforms) are of the form $F(z)=\int_{-\infty}^\infty e^{-(z-t)^2}f(t)\,dt$ or $\int_{-\infty}^\infty e^{-(z-t)^2}dg(t)$. The first type was considered by the reviewer [Ann. of Math. (2) 27, 427–464 (1926)]. The main problem of the author is to find necessary and sufficient conditions which a function $F(z)$ must satisfy in order to admit of such a representation with a generating function $f(t)$ or $g(t)$ having specified properties. These properties will be such that $F(z)$ has derivatives of all orders at $z=0$. Put $a_n=2^{-n}(n!)^{-1}\pi^{-\frac{1}{2}}F^{(n)}(0)$, $F(r, x)=\sum_0^\infty a_nH_n(x)r^n$, where $H_n(x)$ is the nth Hermite polynomial. Necessary and sufficient conditions for representation with an $f(t)\epsilon L_p(-\infty, \infty)$, $p>1$, is (i) $\sum_0^\infty a_n2^{\frac{1}{2}n}r^n$ converges for $r<1$, (ii) $\|F(r, \cdot)\|_p<M$, $0<r<1$. For $p=1$ (ii) is replaced by (iii) $F(r, x)$ converges in the mean of order one to a limit as $r\to 1$. Further (i) and (iv) $\|F(r, \cdot)\|_1\leqq M$ give representation with $g(t)\epsilon BV[-\infty, \infty]$. Here $g(t)$ is never decreasing if and only if $F(r, x)\geqq 0$ and is continuous if and only if $\lim_{r\to 1}(1-r)^{\frac{1}{2}}F(r, x)=0$. Further,

$$f(t)\epsilon L(-\infty, \infty)\cap BV[-\infty, \infty]$$

if and only if (i), (iii), and (v) $\|F'(r, \cdot)\|_1\leqq M$ hold. If $F(z)$ is so representable and (vi) $\int_{-\infty}^\infty \exp(-\frac{1}{2}t^2)|dg(t)|<\infty$, then $g(x)-g(0)=\lim_{r\to 1}\int_0^xF(r, t)\,dt$, for almost all x we have $g'(x)=\lim_{r\to 1}F(r, x)$, and the saltus

$$g(x+0)-g(x-0)=\lim_{r\to 1}[\pi(1-r^2)]^{\frac{1}{2}}F(r, x).$$

The integral equation $F(x)=\int_{-\infty}^\infty e^{(x-t)^2}f(t)\,dt$ has a solution $f(t)$ in $L_2(-\infty, \infty)$ if and only if $\exp(\frac{1}{4}u^2)$ times the Fourier transform of $F(x)$ is in L_2 and this condition gives the author a representation of the solution in terms of Fourier transforms [previously found by the reviewer [loc. cit., p. 428] for the case $F(x)\epsilon L_1\cap L_2$, and in the general case by Doetsch [Math. Z. 41, 283–318 (1936); cf. Tricomi, ibid. 40, 720–726 (1936)]]. In order to derive his various conditions which occupy the third part of the memoir, the author proves a number of theorems concerning singular integrals, Fourier-Stieltjes integrals, summability of Hermitian series, etc. We single out the following. The Fourier-Stieltjes integral of a $g(x)\epsilon BV[-\infty, \infty]$ is summable (C, α), $\alpha>0$, to $g'(x)$ for almost all x as well as by Abel and Weierstrass means. The expression given for $g(x)-g(0)$ above is merely a special case of a theorem on integration of singular integrals which is also applicable to Fourier-Stieltjes integrals with the Fejér, Abel, or Weierstrass kernel. The rest of the prefatory material is devoted to Hermite-Stieltjes and Hermite series. If $g(x)$ is of bounded variation in every finite interval and if (vii) $\int_{-\infty}^\infty e^{-ct^2}|dg(t)|<\infty$ for some $c>0$, then the Hermite-Stieltjes series of $dg(x)$ is summable Abel to $g'(x)$ for almost all x. Here the series may be in terms of $H_n(x)$, normalized $H_n(x)e^{-\frac{1}{2}x^2}$, or, as in theory of probability, $H_n(x)e^{-x^2}$, with corresponding Abel-Hermite kernels $K_i(t, x, r)$, $i=1, 2, 3$. Here

$$\lim_{r\to 1}\int_0^a dx\int_{-\infty}^\infty K_i(t, x, r)\,dg(t)=g(a)-g(0)$$

if $g(t)\epsilon BV[-\infty, \infty]$. Two functions in $BV[-\infty, \infty]$ differ by a constant if and only if their Hermite-Stieltjes coefficients $(i=2)$ are identical. This leads the author to a number of limit theorems for distribution functions. Thus a sequence of distribution functions $\{g_\nu(t)\}$ will converge to a distribution function $g(t)$ at all points of continuity of the latter if and only if the Hermite-Stieltjes coefficients $(i=2)$ of $g_\nu(t)$ converge to those of $g(t)$. If (viii) $\int_{-\infty}^\infty e^{\frac{1}{2}t^2}dg(t)<\infty$ we can take $i=3$ instead. A distribution function is uniquely determined by its Hermite-Stieltjes coefficients of type 3 if (viii) holds. If $g(t)$ is a distribution function whose moments satisfy $\sum 2^{-n}(n!)^{-1}a_{2n}<\infty$ and if $\{g_\nu(t)\}$ is a sequence of distribution functions having moments $a_n^{(\nu)}$ of all orders n, then $\lim a_n^{(\nu)}=a_n$ for all n is necessary and sufficient in order that $g_\nu(t)$ shall converge to $g(t)$ at all points of continuity. If (vii) holds,

$$g(x+0)-g(x-0)=\lim_{r\to 1}[\pi(1-r^2)]^{\frac{1}{2}}\int_{-\infty}^\infty K_i(t, x, r)\,dg(t),$$

$i=1, 2, 3$. If $f(t)e^{-ct^2}\epsilon L(-\infty, \infty)$ for some $c>0$, then

$\lim_{r\to 1}\int_{-\infty}^{\infty} K_i(t, x, r)f(t)dt = f(x)$ almost everywhere. Finally, if $f(t)\epsilon L_p(-\infty, \infty)$, $1\leqq p$, then $\int_{-\infty}^{\infty} K_i(t, x, r)f(t)\,dt$ converges to $f(x)$ in the mean of order p. *E. Hille.*

Stone, M. H. The algebraization of harmonic analysis. Math. Student **17**, 81–92 (1949).

This is an excellent expository paper delivered in connection with the Symposium on Harmonic Analysis organized by the Sixteenth Biennial Conference of the Indian Mathematical Society, and held at Madras on December 27, 1949. Wiener's Tauberian theorems are discussed together with their algebraic formulations as given by Gelfand and Segal. The following topics are also discussed: Weyl's theory of almost periodic functions; the Peter-Weyl theory of compact Lie groups; the Pontrjagin duality theorem; the Stone characterization of the algebra of bounded continuous functions; and Schwartz's theory of distributions. *N. Dunford.*

Levitan, B. M. Some questions of the theory of almost periodic functions. Amer. Math. Soc. Translation no. **28**, 53 pp. (1950).

Translated from Uspehi Matem. Nauk (N.S.) **2**, no. 6(22), 174–214 (1947); these Rev. **10**, 293.

Petersen, Richard. On Heaviside's expansion theorem. Mat. Tidsskr. B. **1950**, 82–85 (1950). (Danish)

Heaviside's expansion theorem asserts that the Laplace transform of an exponential series $f(t)\sim\sum_1^\infty c_n e^{s_n t}$ is given by $\sum_1^\infty c_n(s-s_n)^{-1}$. The author shows that this formula is actually valid for $\Re(s)>0$ if $f(t)\sim\sum_1^\infty A_n e^{i\Lambda_n t}$ is almost periodic provided zero is not in the closure of the set $\{\Lambda_n\}$ and $\sum|A_n||\Lambda_n|^{-1}<\infty$. If merely $\sum_1^\infty|A_n||\Lambda_n|^{-p}$ converges, p a positive integer >1, then a polynomial of degree $(p-2)$ should be subtracted from each term $A_n(s-s_n)^{-1}$, the polynomial being determined, though not uniquely, by the relation between the Laplace transforms of a function and of its derivative. [Reviewer's note. The series converges also for $\Re(s)>0$ and in some neighborhood of the origin which is not a singular point. For the Laplace transform of almost periodic functions cf. Bochner and Bohnenblust, Ann. of Math. (2) **35**, 152–161 (1934).] *E. Hille.*

Maravall, Dario. The nonconvergence of the Fourier integral and discontinuity waves in physics. Revista Mat. Hisp.-Amer. (4) **10**, 77–81 (1950). (Spanish)

Polynomials, Polynomial Approximations

Ibragimov, I. I. On the best approximation by polynomials of the functions $[ax+b|x|]|x|^s$ on the interval $[-1, +1]$. Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 405–412 (1950). (Russian)

Denote the quantity described in the title by $E_n(s, a, b)$, $s>-1$. The author shows that

$$\lim_{n\to\infty} n^{s+1}E_n(s, a, b)=\lambda(s+1, a, b),$$

where

$$1-s^{-1}<\frac{\pi\lambda(s+1, a, b)}{\omega(s, a, b)\Gamma(s+1)}<1,$$

$$\omega(s, a, b)=\{a^2\sin^2\tfrac{1}{2}\pi s+b^2\cos^2\tfrac{1}{2}\pi s\}^{\frac{1}{2}}.$$

R. P. Boas, Jr. (Evanston, Ill.).

Merli, Luigi. Sull'approssimazione delle funzioni continue di due variabili mediante polinomi. Boll. Un. Mat. Ital. (3) **5**, 68–71 (1950).

Pour les fonctions de deux variables, définis dans le carré $(0\leqq x\leqq 1,\ 0\leqq y\leqq 1)$, et lipschitziennes d'ordre 1 en x et y, la suite des polynomes d'interpolation, formés à partir des polynomes de Tchebicheff, de degré m en x, et de degré n en y, converge vers la fonction lorsque le rapport m/n, et son inverse, sont bornés. *J. Favard* (Paris).

Nikol'skiĭ, S. M. On the best approximation of differentiable, nonperiodic functions by means of polynomials. Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 185–197 (1950). (Russian)

This paper consists of four sections and gives complete proofs of results announced earlier [C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 95–98, 191–194 (1947); Doklady Akad. Nauk SSSR (N.S.) **58**, 25–28, 185–188 (1947); these Rev. **9**, 90, 280, 282]. *A. Zygmund* (Chicago, Ill.).

Geronimus, J. Sur quelques transformations des fractions continues et les systèmes correspondants des polynomes orthogonaux. Akad. Nauk Ukrain. RSR. Zbirnik Prac' Inst. Mat. **1946**, no. 8, 121–133 (1947). (Ukrainian. Russian and French summaries)

For given polynomials $u(z)$ and $v(z)$, of degrees k and $k-1$ respectively, each having real distinct zeros, and for a given real J-fraction in $u=u(z)$, the author constructs a J-fraction in z whose approximants of order nk are the approximants of order n for the original J-fraction. A simple set of conditions on $u(z)$, stated in terms of the zeros of $v(z)$, insures that the constructed J-fraction is real. A relation between the Stieltjes distributions of the two J-fractions is obtained, and the case $k=2$ is considered in more detail.

W. T. Scott (Evanston, Ill.).

Davis, Philip. An application of the theory of basic series to theorems of Bernstein-Widder type. Amer. J. Math. **72**, 787–791 (1950).

Let $\{p_n(z)\}$ be a set of polynomials which is basic, in the terminology of J. M. Whittaker. The author gives conditions under which $f(x)$ is necessarily absolutely monotonic on an interval if $p_n(d/dx)f(x)\geqq 0$ there. The conditions are fulfilled, for example, by Lidstone polynomials and by an extensive class of Appell polynomials which includes the Hermite polynomials. Similarly $f(x)\geqq 0$, $(-1)^n(D^2+a_1^2)\cdots(D^2+a_n^2)f(x)\geqq 0$ with $D=d/dx$, a_k real, implies that $f(x)$ is completely convex, i.e., $(-1)^n f^{(2n)}(x)\geqq 0$.

R. P. Boas, Jr. (Evanston, Ill.).

Makar, Ragy, et Makar, Bushra H. Sur la base somme de bases de polynomes. Bull. Sci. Math. (2) **74**, 138–145 (1950).

Let $\{p_n(z)\}$ and $\{q_n(z)\}$ be basic sets of polynomials; let $u_n(z)=p_n(z)+q_n(z)$; then $\{u_n(z)\}$ is not necessarily basic, but is so if the original sets are simple and have leading coefficient one [are monic, in the authors' terminology]; even in this case $\{u_n\}$ is not necessarily effective in $|z|\leqq R$ if both $\{p_n\}$ and $\{q_n\}$ are. [For terminology see J. M. Whittaker, Sur les séries de base de polynomes quelconques, Gauthier-Villars, Paris, 1949; these Rev. **11**, 344.] The authors call $\{p_n\}$ algebraic if its matrix satisfies a polynomial equation. They prove the following theorems. (1) If $\{p_n\}$ and $\{q_n\}$ are simple, monic, and effective in $|z|\leqq R$,

$\lambda+\mu=1$, $u_n=\lambda p_n+\mu q_n$, and $\{u_n\}$ is algebraic, then $\{u_n\}$ is effective in $|z|\leqq R$. (2) If $\{p_n\}$ is simple, monic, and algebraic, $\lambda+\mu=1$, the sets $\{p_n(z)\}$ and $\{\lambda p_n(z)+\mu z^n\}$ are effective in the same region. (3) If $\{p_n\}$ is simple, monic, and algebraic, and $|\pi_{ni}|$ or $|p_{ni}|\leqq k\rho^{n-i}$, $k\geqq 1$, then $\{p_n\}$ is effective in $|z|\leqq\rho$; this is not necessarily true if "algebraic" is omitted. *R. P. Boas, Jr.* (Evanston, Ill.).

Vučkič, Milenko. On Čebyšev polynomials. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **4**, 209–220 (1949). (Serbo-Croatian)

Expository article. *R. P. Boas, Jr.* (Evanston, Ill.).

Delange, Hubert. Sur certains polynomes introduits par Tchebichef. C. R. Acad. Sci. Paris **231**, 602–604 (1950).

A classical problem of Tchebychev consists of determining the abscissas x_i and the constant H such that the quadrature formula $\int_0^1 f(x)\,dx=H\sum_{i=1}^n f(x_i)$ should hold for all polynomials of degree n. This leads to $H=2/n$ and the x_i must be the roots of a certain polynomial $P_n(z)$ which arises as follows. Let $\varphi(z)=\exp\{\frac12(z+1)\log(z+1)-\frac12(z-1)\log(z-1)-1\}$, where $\log(z\pm1)$ are real when $z>1$. Then for $|z|>1$ an expansion of the form $[\varphi(z)]^n=P_n(z)+\sum_{j=1}^\infty \alpha_j^{(n)}z^{-j}$ holds. The principal question is whether the roots x_i are on $(-1,1)$. This is not the case in general as it was shown by direct calculation for $n=8$, $10\leqq n\leqq 21$. The author studies the asymptotic distribution of the x_i as n is large. It is shown that these numbers cluster about a curve which can be characterized by the condition $|\varphi(z)|=2/e$. Other characterizations are also given based on potential-theoretic interpretations. *G. Szegö* (Stanford University, Calif.).

Sansone, Giovanni. La formula di approssimazione asintotica dei polinomi di Tchebychef-Laguerre col procedimento di J. V. Uspensky. Math. Z. **53**, 97–105 (1950).

The author uses the integral representation of the Laguerre polynomials $L_n^{(\alpha)}(x)$ due to Uspensky [see Szegö, Orthogonal polynomials, Amer. Math. Soc. Colloquium Publ., v. 23, New York, 1939, (5.6.5); these Rev. **1**, 14] which involves the Hermite polynomials. He obtains the following asymptotic formula:

$$L_n^{(\alpha)}(x)=\frac{\Gamma(n+\alpha+1)}{n!}\frac{2^{\alpha+}e^{\frac12 x}\pi^{-\frac12}}{(Nx)^{\frac12\alpha+\frac14}}\Big\{\cos[(Nx)^{\frac12}-(\tfrac12\alpha+\tfrac14)\pi]$$
$$+\frac{\frac18x^2-(\alpha+\frac12)x-\frac13\alpha^2+\frac18}{(Nx)^{\frac12}}\sin[(Nx)^{\frac12}-(\tfrac12\alpha+\tfrac14)\pi]$$
$$+\frac{R(n,x,\alpha)}{Nx}\sum_{k=0}^{3l+3}q_k(\alpha)x^{\frac12 k}\Big\}.$$

Here $\alpha>-\frac12$, $N=4n+1$, $1\leqq Nx\leqq N^{\frac13-2\epsilon}$, $0<\epsilon<\frac16$, $l\geqq l_0(\alpha,\epsilon)$, $|R|\leqq1$; the constants $q_k(\alpha)$ are nonnegative and they depend only on α. For l_0 an explicit formula is given. *G. Szegö* (Stanford University, Calif.).

Cooper, R. The extremal values of Legendre polynomials and of certain related functions. Proc. Cambridge Philos. Soc. **46**, 549–554 (1950).

Let $P_n(x)$ be the nth Legendre polynomial. Denoting the rth maximum of $|P_n(x)|$ (counting in decreasing order from $x=1$) by $\mu_n^{(r)}$ the author proves that $\mu_n^{(r)}$ is decreasing when n increases provided n is sufficiently large. [In the meantime the reviewer has proved this for $n\geqq r+1$; see the following review.] Similar questions are considered for the Jacobi polynomials and for the associated functions. *G. Szegö* (Stanford University, Calif.).

Szegö, G. On the relative extrema of Legendre polynomials. Boll. Un. Mat. Ital. (3) **5**, 120–121 (1950).

Todd, John. On the relative extrema of the Laguerre orthogonal functions. Boll. Un. Mat. Ital. (3) **5**, 122–125 (1950).

Szász, Otto. On the relative extrema of ultraspherical polynomials. Boll. Un. Mat. Ital. (3) **5**, 125–127 (1950).

A study of numerical tables of orthogonal polynomials led Todd to a conjecture about the relative extrema of some such polynomials. R. Cooper [see the preceding review] proved that Todd's conjecture for Legendre polynomials holds at least asymptotically for large n. In the three notes under review more precise results are proved. Szegö proves that the rth relative maximum of the numerical value of the Legendre polynomial of degree n, is a decreasing function of n (r fixed). Let $L_n(x)$ be the Laguerre polynomial of degree n, set $\phi_n(x)=e^{-\frac12 x}L_n(x)$, and let $\mu_{r,n}$ be the rth relative extremum of $\phi_n(x)$. Todd proves that for fixed odd (even) r, $\mu_{r,n}$ is a monotonic increasing (decreasing) function of n. Let $\mu_{r,n}(\lambda)$ be the rth relative extremum of the ultraspherical polynomial $P_n^\lambda(x)$. Szász proves that for fixed positive λ and fixed r, $n!|\mu_{r,n}(\lambda)|/\Gamma(n+2\lambda)$ is a monotonic decreasing function of n. *A. Erdélyi.*

Gel'fond, A. O. On the generalized polynomials of S. N. Bernštein. Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 413–420 (1950). (Russian)

Let $0=\alpha_0<\alpha_1\leqq\alpha_2\leqq\cdots$, $\alpha_n\to\infty$, $\sum\alpha_n^{-1}=\infty$. If

$$\tau_{kn}=[(1-\alpha_1/\alpha_{k+1})\cdots(1-\alpha_1/\alpha_n)]^{1/\alpha_1}$$

and

$$q_{kn}=(-1)^{n-k}\frac{\alpha_{k+1}\cdots\alpha_n}{2\pi i}\int_{|z|=1+\alpha_n}\frac{x^z\,dz}{(z-\alpha_k)\cdots(z-\alpha_n)}$$
$$=(-1)^{n-k}\alpha_{k+1}\cdots\alpha_n[\alpha_k\cdots\alpha_n],\qquad 0\leqq k\leqq n,$$

where $[\alpha_k\cdots\alpha_n]$ are the divided differences of the function x^z, then the "polynomial" $B_n(f,x)=\sum_{k=0}^n f(\tau_{kn})q_{kn}(x)$ converges uniformly towards $f(x)$ for any $f(x)$ continuous on $[0,1]$. The $q_{kn}(x)$ are linear aggregates of functions $x^{\alpha_k}\log^\nu x$, where ν is a nonnegative integer less than the multiplicity μ_k of α_k. In case $\mu_k=1$, $k=1,2,\cdots$, this has been given by Hirschman and Widder [Duke Math. J. **16**, 433–438 (1949); these Rev. **11**, 29]. The author uses a more convenient technique which enables him to estimate the degree of approximation. Two inaccuracies on pp. 417–418 are easily corrected. *G. Lorentz* (Toronto, Ont.).

Obrechkoff, N. Sur les zéros des polynomes et de quelques fonctions entières. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **37**, 1–115 (1941). (Bulgarian. French summary)

"Ce travail contient une serie des résultats nouveaux pour les zéros des polynomes et de quelques classes importantes des fonctions entières, qui étaient l'objet des recherches diverses dans la mathématique modèrne. Différentes questions ouvertes sont résolues. D'abord nous considérons des fonctions entières limites des polynomes dont les zéros sont tous réels. Ensuite nous s'occupons avec les zéros de la fonction (1) $F(z)=c_1e^{a_1z}+\cdots+c_ne^{a_nz}$, où $a_1,\cdots,a_n$ sont des nombres complexes arbitraires différents et $c_1,\cdots,c_n$ sont des nombres complexes différents de zéro. Nous avons donné quelques parties de ce travail dans quelques publications [C. R. Acad. Sci. Paris **206**, 1874–1876 (1938); **208**, 74–76,

1270–1272 (1939); Mathematica, Cluj 15, 165–173 (1939); Math. Z. 45, 747–750 (1939); Comment. Math. Helv. 12, 66–70 (1939); Quelques classes de fonctions entières . . . , Actualités Sci. Ind., no. 891, Hermann, Paris, 1941; ces Rev. 1, 49, 193; 7, 516]." *From the author's summary.*

Sanielevici, S. On the problem of Hurwitz. Acad. Repub. Pop. Române. Bul. Şti. A. 1, 543–550 (1949). (Romanian. Russian and French summaries)

This paper is concerned with the Hurwitz criterion for the solution of the question that an algebraic equation $f(x)=0$ with real coefficients have only roots with negative real parts. Here the problem is solved by the formation of a new equation with roots $\frac{1}{2}(x_i+x_k)$, where x_i, x_k are roots of $f(x)=0$. The new equation has only roots with negative real parts if its coefficients are positive. The solution to the Hurwitz problem can also be found by the substitution $x=\xi+i\eta$, from which $f(\xi+i\eta)=P(\xi,\eta)+i\eta Q(\xi,\eta)$ is obtained, and the elimination of η between the equations $P(\xi,\eta)=0$ and $Q(\xi,\eta)=0$. Thus a connection is established between the proposed method and the Hurwitz criterion.

E. Frank (Chicago, Ill.).

Verblunsky, S. A theorem on quartic polynomials. Duke Math. J. 17, 507–510 (1950).

In this paper it is shown that the real quartic polynomial $f(x)=x^4+ax^3+bx^2+cx+1$ satisfies the condition $f(x)>0$, for $x>0$, in one of the following mutually exclusive cases, and only in such cases: (i) $a>0$, $c>0$, $b+2+2(ac)^{\frac{1}{2}}>0$; (ii) $a>0$, $c>0$, $b+2+2(ac)^{\frac{1}{2}}\leqq 0$, $L=(n-4)(b+2n-2)^{\frac{1}{2}}-(a+c)<0$, where $3n=6-b+d^{\frac{1}{2}}$, $d=12+b^2-3ac$; (iii) $a<0$, $c<0$, $L<0$, $M(b)=b^2+20b-28-(b-6)|b-6|>8ac, b+2>0$; (iv) $ac\leqq 0$, $L<0$. *E. Frank* (Chicago, Ill.).

Bothwell, Frank E. Nyquist diagrams and the Routh-Hurwitz stability criterion. Proc. I.R.E. 38, 1345–1348 (1950).

The author gives very brief descriptions of the Nyquist and Routh-Hurwitz methods for investigating the stability of linear systems, and argues that the latter method is superior in some cases. Most of the specific statements made in the paper are unexceptionable, but the reviewer is of the opinion that the true situation has not been considered very deeply. *L. A. MacColl* (New York, N.Y.).

Teodorčik, K. F. The trajectories of the roots of the characteristic equation of a system of the third order with continuous variation of the free member and the maximum stability thereby attainable. Akad. Nauk SSSR. Žurnal Tehn. Fiz. 18, 1394–1398 (1948). (Russian)

The location of the roots of quadratic and cubic equations is discussed with a view to the application to stability problems. *R. Bellman* (Stanford University, Calif.).

Bückner, Hans. A variational problem for the roots of a cubic equation. (A contribution to the theory of servomechanisms.) Quart. Appl. Math. 8, 293–296 (1950).

For a servo-mechanism with the characteristic equation $x^3+a_1x^2+a_2x+a_3=0$ the author defines ξ as the minimum real part among the negatives of the three roots. Then ξ is a rough measure of the stability and performance of the servo. The author derives the value of a_3 that maximizes ξ, but he does not show that variation of a_3 alone is physically significant. *G. R. Stibitz* (Burlington, Vt.).

Special Functions

Parodi, Maurice. Sur quelques applications d'un théorème de Laguerre-Polya. C. R. Acad. Sci. Paris 231, 889–890 (1950).

The theorem referred to in the title asserts that the number of changes of sign of $f(t)$ in $(0,\infty)$ is an upper bound for the number of zeros of $\varphi(s)=\int_0^\infty e^{-st}f(t)\,dt$ in the interval of convergence. The author takes $f(t)=e^{-x_0t}P_m(t)$, where x_0 is real and $P_m(t)$ is any polynomial, and considers the Laplace transforms of $t^nf(t^2)$, $f(e^t-1)$, $t^re^{-t}P_m(1/t)$, all of which can be expressed as integrals. Upper bounds for the number of zeros of the functions defined by these integrals follow. *A. Erdélyi* (Pasadena, Calif.).

Lebedev, N. N. Some integral representations for products of sphere functions. Doklady Akad. Nauk SSSR (N.S.) 73, 449–451 (1950). (Russian)

Representations for products of associated Legendre functions, of the form $P_n^m(\cosh\alpha)P_n^{-m}(\cosh\alpha')$ and $Q_n^m(\cosh\alpha)P_n^{-m}(\cosh\alpha')$ are given in terms of integrals whose integrands involve a product of a Legendre function of order $(m-\frac{1}{2})$ and an exponential or hyperbolic function of $(n+\frac{1}{2})\psi$, ψ being the variable of integration.

J. L. B. Cooper (Cardiff).

Robin, Louis. Développements en séries entières des fonctions de Legendre et associées de Legendre, au voisinage de chacun des points singuliers ±1. C. R. Acad. Sci. Paris 231, 746–748 (1950).

Generalized Legendre functions are defined, and can be represented in many ways, by hypergeometric series. For certain integer values of the parameters the series appear in indeterminate form (logarithmic case). The author computes several of these logarithmic cases. *A. Erdélyi.*

Toscano, Letterio. Funzione generatrice dei prodotti di polinomi di Laguerre con gli ultrasferici. Boll. Un. Mat. Ital. (3) 5, 144–149 (1950).

The author expresses the sum of the series

$$\sum_{n=0}^{\infty}\frac{z^nn!}{\Gamma(2\alpha+1+n)}L_n^{(2\alpha)}(x)C_n^{\alpha+\frac{1}{2}}(y),$$

in which $L_n^{(2\alpha)}$ is the generalized Laguerre polynomial and C_n^{ν} is the Gegenbauer polynomial, in closed form. He discusses some particular cases, and some related expansions.

A. Erdélyi (Pasadena, Calif.).

Mayol, Guillermo. Hypergeometric series. Gaceta Mat. (1) 2, 115–126 (1950). (Spanish)

Expository paper on ${}_{k+1}F_k(1, a_1, \cdots, a_k; b_1, \cdots, b_k; x)$.

A. Erdélyi (Pasadena, Calif.).

Lauwerier, H. A. The use of confluent hypergeometric functions in mathematical physics and the solution of an eigenvalue problem. Appl. Sci. Research A. 2, 184–204 (1950).

In some physical problems the zeros of Whittaker's confluent hypergeometric function $M_{k,m}(\omega)$ are of considerable interest. For the computation of large zeros, the author obtains asymptotic expansions. His first method is based on a suitable expansion of the integrand. This part of the paper appears to be related to some work by Tricomi [Ann. Mat. Pura Appl. (4) 26, 141–175 (1947); 28, 263–289 (1949); Comment. Math. Helv. 22, 150–167 (1949); these Rev. 10, 605; 12, 96; 10, 703]. The second method is based on the

method of steepest descents and is given in more detail in another paper by the author [Nederl. Akad. Wetensch., Proc. 53, 188–195 = Indagationes Math. 12, 26–33 (1950); these Rev. 11, 593]. The third method is similar to the first one. Small roots are computed from the power series.
A. Erdélyi (Pasadena, Calif.).

Kline, Morris. A Bessel function expansion. Proc. Amer. Math. Soc. 1, 543–552 (1950).

Let the function G be defined by

$$G(a, k; \nu, R) = J_\nu(kR)N_\nu(kR+ka) - J_\nu(kR+ka)N_\nu(kR),$$

in which J_ν and N_ν are the Bessel and Neumann functions of order ν. In a certain electromagnetic waveguide problem an approximate expression for G is required under the conditions that R and ν are large and ν/R is constant ($=d$). The author derives a Taylor's expansion of G about $R=\infty$, the first two terms of which are given explicitly, viz.,

$$G(a, k; dR, R)$$
$$= \frac{2}{\pi}\left[\frac{a}{R}\frac{\sin x}{x} + \frac{a^2}{2R^2}\left\{\frac{\sin x}{\gamma x} - \left(1+\frac{1}{\gamma}\right)\cos x\right\} + O\left(\frac{1}{R^3}\right)\right],$$

where $x^2 = -k^2a^2\gamma$, $\gamma = d^2/k^2 - 1$. This result is discussed in terms of the waveguide problem. [Reviewer's remark: It is easy to show that $G(a, k; dR, R)$ is a one-valued analytic function of R if $|R| > |a|$, but G is not one-valued if $|R| < |a|$. The author's Taylor expansion, therefore, is divergent if $|a/R| > 1$, contrary to the author's statement. The error in his analysis comes from the fact that equation (4) holds only if $|\alpha/\beta| < 1$.]
C. J. Bouwkamp.

Differential Equations

★Petrovskiĭ, I. G. Lekcii po teorii obyknovennyh differencial'nyh uravneniĭ. [Lectures on the Theory of Ordinary Differential Equations]. 3d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1949. 208 pp.

Part I. A single differential equation of the 1st order with one unknown function: General notions; The simplest differential equations; General theory. Part II. Systems of ordinary differential equations: General theory; General theory of linear systems; Linear systems with constant coefficients. Appendix: Partial differential equations of the 1st order in one unknown function.
Table of contents.

Zwirner, Giuseppe. Criteri di unicità per gli integrali delle equazioni differenziali del primo ordine. Rend. Sem. Mat. Univ. Padova 19, 273–293 (1950).

This paper gives five new theorems concerning the uniqueness of the solution of a differential equation $y' = f(x, y)$ and an initial condition $y(x_0) = y_0$. The sets of sufficient conditions for the uniqueness of the solution are complicated and of great generality, and they include, as special cases, most of the sets of sufficient conditions that are to be found in the literature.
L. A. MacColl (New York, N. Y.).

Yoshizawa, Taro, and Hayashi, Kyuzo. On the uniqueness of solutions of a system of ordinary differential equations. Mem. Coll. Sci. Univ. Kyoto Ser. A. 26, 19–29 (1950).

Let $dy_i/dx = f_i(x, y_1, \cdots, y_n)$, $i = 1, \cdots, n$, be a system of ordinary differential equations which for $0 \leqq x \leqq a$ admits the solution $y_1 = \cdots = y_n = 0$. Let S_0 and S_1 be neighborhoods of the points $x=0$, $y_i=0$ and $x=a$, $y_i=0$, respectively. Then a necessary and sufficient condition is given for the above solution $y_1 = \cdots = y_n = 0$ to be the only one which passes through points of S_0 and S_1. The proof is based on a generalization of a function introduced by Okamura [same Mem. Ser. A. 23, 225–231 (1941); these Rev. 7, 442] in connection with the treatment of uniqueness questions. Applications to differential equations of higher order and some generalizations are given.
E. H. Rothe (Ann Arbor, Mich.).

Miller, Kenneth S. On iterative methods in linear differential equations. Trans. Amer. Math. Soc. 69, 195–207 (1950).

This paper considers a linear homogeneous differential equation of the nth order $Lu=0$ with coefficients continuous on the finite interval $0 \leqq x \leqq c$ and linear nonhomogeneous boundary conditions of the usual type. Approximate solutions u_m, $m = 0, 1, \cdots$, and errors $\epsilon_m = Lu_m$ are defined. Under certain restrictions on the boundary conditions and the interval, it is shown that for m infinite u_m approaches a solution of the differential system. This is accomplished by setting up a self-adjoint finite normed operator T such that $\epsilon_{m+1} = T\epsilon_m$ and $\lim \epsilon_m = 0$. There is some discussion of the use of machines to perform the calculations.
J. M. Thomas (Durham, N. C.).

Coe, C. J. The generalized Leibniz formula. Amer. Math. Monthly 57, 459–466 (1950).

de Castro Brzezicki, Antonio. On small oscillations of dissipative systems. Revista Mat. Hisp.-Amer. (4) 10, 47–50 (1950). (Spanish)

This is an elementary discussion of the oscillations of a dynamical system, with two degrees of freedom, governed by a system of differential equations of motion which are linear with constant coefficients. The few results obtained can hardly be regarded as new.
L. A. MacColl.

Prodi, Giovanni. Un'osservazione sugl'integrali dell'equazione $y'' + A(x)y = 0$ nel caso $A(x) \to +\infty$ per $x \to \infty$. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 8, 462–464 (1950).

The author proves the following theorem: If

$$\lim_{x\to+\infty} A(x) = \infty,$$

and $A(x)$ is nondecreasing, but not otherwise restricted, then the differential equation $y'' + A(x)y = 0$ possesses at least one solution $y_1(x)$ such that $\lim_{x\to+\infty} y_1(x) = 0$.
W. Wasow (Cambridge, Mass.).

The result of the paper was quoted earlier by H. Milloux (Prace Mat.-Fiz. 41, 21–54 (1934))

Shtokalo, J. On the theory of linear differential equations with quasi periodic coefficients. Akad. Nauk Ukrain. RSR. Zbirnik Prac' Inst. Mat. 1946, no. 8, 163–176 (1947). (Ukrainian. Russian and English summaries)

The author considers the nth order differential equation $x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x = 0$, where $a_k(t) = a_k{}^0 + \epsilon f_k(t)$, where $a_k{}^0$ is a constant and $f_k(t)$ are of the form $\sum A_\nu{}^{(k)}e^{i\nu t}$ summed over a finite range of ν. The equation with $\epsilon = 0$ is assumed to have characteristic roots with negative real parts which are all distinct. Formal series solutions are obtained and their asymptotic character demonstrated over $0 \leqq t \leqq \infty$. The formal solutions are of the form $\xi(t)e^{\rho t}$, where $\rho = \rho_0 + \epsilon\rho_1 + \epsilon^2\rho_2 + \cdots$ and $\xi(t) = 1 + \epsilon\xi_1(t) + \epsilon^2\xi_2(t) + \cdots$ where the ρ_i are constants and the $\xi_i(t)$ are of the same general form as the $f_k(t)$.
N. Levinson (Cambridge, Mass.).

Ryabov, B. A. Auto-oscillations in some servo-systems restrained by the presence of damping (Coulomb) friction. Doklady Akad. Nauk SSSR (N.S.) 73, 283–286 (1950). (Russian)

The system $\sum P_{k,m}(p)x_k=0$ ($m=1, \cdots, n-1$, and $\sum$ ranging over $1\leqq k\leqq n$) together with $\sum P_{k,n}(p)x_k=-F_0 \operatorname{sgn}(px_n)$ is considered, where the $P_{k,m}$ are all polynomials in p of degree 2 and $p=d/dt$. Another case is also considered.

N. Levinson (Cambridge, Mass.).

Obi, Chike. Subharmonic solutions of non-linear differential equations of the second order. J. London Math. Soc. 25, 217–226 (1950).

The differential equation considered is

$$\ddot{x}+a^2x=\epsilon\phi_1(t)+k\phi_2(t)+\epsilon\phi_3(x, \dot{x})+k\phi_4(x, \dot{x})+\phi_5(x, \dot{x}, k, \epsilon, t),$$

where the right member is analytic in all variables and ϕ_5 does not contain terms in ϵ and k below the second degree. The right member is periodic in t. The parameters ϵ and k are small. Sufficient conditions are given for the existence of a range of values of k as a function of ϵ, $K(\epsilon)$, such that when k lies in one of these ranges the equation has subharmonic solutions. When $\epsilon\rightarrow 0$ the ranges of k, $K(\epsilon)$, also tend to $k=0$. *N. Levinson* (Cambridge, Mass.).

Bothwell, Frank E. Transients in multiply periodic nonlinear systems. Quart. Appl. Math. 8, 247–254 (1950).

Systems of the form $\sum_{j=1}^{n}(a_{ij}\ddot{y}_j-b_{ij}y_j)+\mu g_i=0, i=1, \cdots, n$, are considered, where the a's and b's are constants, μ is a small parameter, and the g_i are functions of μ, t, the y_j, and certain derivatives; it is assumed that the g_i are sums of periodic functions of t having several different periods $\Omega_1, \cdots, \Omega_m$. A method which generalizes slightly those of Kryloff and Bogoliuboff [Introduction to Non-linear Mechanics, Princeton University Press, 1943; these Rev. 4, 142] is applied to determine the periodic solutions "in the first approximation" (i.e., for $\mu\rightarrow 0$) and their stability properties. An electric circuit with two degrees of freedom is studied as an application. *J. L. Massera.*

Bromberg, P. V. On the problem of stability of a class of nonlinear systems. Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 561–562 (1950). (Russian)

The author obtains a slight extension of the stability criteria given by Lur'e [same journal 12, 651–666 (1948); these Rev. 11, 110] for the system $\dot{x}_s=\lambda_s x_s+f(\sigma), s=1, \cdots, n$, $\dot{\sigma}=\sum_1^n\beta_a x_a-rf(\sigma)$. In particular, the author is able to treat the case when $r+\sum_1^n\beta_a/\lambda_a=0$, to which the Lur'e method is inapplicable. *J. G. Wendel* (New Haven, Conn.).

Al'muhamedov, M. I. On conditions for the existence of stable and unstable centers. Amer. Math. Soc. Translation no. 34, 8 pp. (1950).

Translated from Doklady Akad. Nauk SSSR (N.S.) 67, 961–964 (1949); these Rev. 11, 110.

***Bulgakov, B. V. Kolebaniya. Tom I.** [Oscillations. Vol. I.]. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1949. 464 pp.

Elements of matrix calculus; Elements of operational calculus; Equations of analytic dynamics; Statement of the problem, linear systems; Free oscillations of nonlinear systems; Forced oscillations of nonlinear systems.

Table of contents.

Bulgakov, B. V. The discriminant curve and the domain of aperiodic stability. Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 453–458 (1950). (Russian)

The linear system $dX/dt=(\mu A+\nu B+C)X$, where $X=X(t)$ is a vector, A, B, C are real constant matrices, and μ, ν are real parameters, has for its characteristic polynomial an expression of the form $\Delta(z)=P(z)\mu+Q(z)\nu+R(z)$. The domain of aperiodic stability is the region in the (μ, ν)-plane in which $\Delta(z)=0$ has all its roots real and negative. The discriminant curve Y is useful for the study of the domain of aperiodic stability; Y is given in parametric form by the equations $P(\epsilon)\mu+Q(\epsilon)\nu+R(\epsilon)=0=P'(\epsilon)\mu+Q'(\epsilon)\nu+R'(\epsilon)$, where ϵ is a real parameter. If a point (μ, ν) crosses Y in an "appropriate direction" a pair of complex roots of $\Delta(z)=0$ disappears. The main theorem characterizes this direction precisely. *J. G. Wendel* (New Haven, Conn.).

Levinson, Norman. Small periodic perturbations of an autonomous system with a stable orbit. Ann. of Math. (2) 52, 727–738 (1950).

The differential equation $dx/dt=X(x)+\epsilon R(t, x, \epsilon)$ for the n-dimensional vector $x=x(t)$ is assumed to possess, for $\epsilon=0$, a periodic solution $x=f(t)$ of period 2π which is stable in the strong sense. The perturbation term $\epsilon R(t, x, \epsilon)$ is taken to be periodic in t with period T. The functions $X(x)$ and $R(t, x, \epsilon)$ are supposed to be of class C^3 with respect to x and continuous with respect to t and ϵ. The author proves that there exists, for small ϵ, a closed contour C_ϵ near $x=f(t)$ such that any solution of the differential system that is on C_ϵ for $t=0$ will again be on C_ϵ for $t=T$. The totality of these solutions generates a two-dimensional surface H_ϵ in $(n+1)$-dimensional (x, t)-space. This surface is shown to be stable in the sense that solutions of the differential equation near H_ϵ will tend to H_ϵ as $t\rightarrow\infty$. With the help of the equation of the surface H_ϵ the study of those solutions of the differential system that lie on H_ϵ can be reduced to the theory of a differential equation on a torus. It then follows that a solution on H_ϵ is periodic or ergodic, depending on whether a certain constant $\mu(\epsilon)$ is rational or irrational.

W. Wasow (Cambridge, Mass.).

Niemytski, V. Intégration qualitative du système

$$dx/dt=Q(x, y); \quad dy/dt=P(x, y)$$

au moyen de réseaux universels de lignes polygonales. Učenye Zapiski Moskov. Gos. Univ. 100, Matematika, Tom I, 34–52 (1946). (Russian. French summary)

The author extends certain aspects of an earlier paper [Rec. Math. [Mat. Sbornik] N.S. 16(58), 307–344 (1945); these Rev. 7, 298]. The complexity of the results precludes stating them here but they are given in the adequate French summary. *N. Levinson* (Cambridge, Mass.).

MacColl, L. A. Pseudo closed trajectories in the family of trajectories defined by a system of differential equations. Quart. Appl. Math. 8, 255–263 (1950).

A pseudo-closed trajectory (p.c.t.) of a system of differential equations of the form $dx/dt=X(x, y)$, $dy/dt=Y(x, y)$ is a simple closed oriented curve consisting of a finite number of arcs of oriented trajectories. The points of a p.c.t. where two such arcs meet are intersections of the curves $X(x, y)=0$ and $Y(x, y)=0$. These intersections are supposed to be simple. They are called interior vertices of the p.c.t. if the two branches of $X=0$ and $Y=0$ that are not part of the p.c.t. penetrate there into the interior of the p.c.t. The author proves that the numbers N_n, N_f, N_e, N_s

of those nodes, foci, centers, and saddle points, respectively, of the system that lie inside a p.c.t. are related to the number N_{iv} of its interior vertices by the equality $N_n+N_f+N_c-N_s=1+N_{iv}$. As an application the topological structure of the family of trajectories inside a p.c.t. with $N_{iv}=1$ or 2, $N_c=0$, $N_s=0$, and without closed trajectories there, is analyzed completely. *W. Wasow.*

Barbašin, E. A. Dispersive dynamical systems. Uspehi Matem. Nauk (N.S.) **5**, no. 4(38), 138–139 (1950). (Russian)

In this paper several theorems on regular families of curves filling an open subregion G of an n-dimensional manifold M and on first order partial differential equations are stated without proof. The curves are assumed defined by differential equations $x_i=X_i(x_1, \cdots, x_n)$, $i=1, \cdots, n$, and are said to form a dispersive system if they are homeomorphic to a family of parallel lines. Theorem 1. The system is dispersive if and only if there exists a function u such that $\sum(X_i\partial u/\partial x_i)$ is positive and bounded away from 0. The system is called unstable if every curve has no α or ω limit points in G; it is said to have no improper saddle point if, whenever a sequence of arcs p_nq_n of the curves are such that p_n converges to p in G, q_n converges to q in G, then every sequence formed of points r_n between p_n and q_n on p_nq_n has a limit point in G. Theorem 2. If the system of curves is unstable and has no improper saddle point, then there exists a solution v of the differential equation $\sum(X_i\partial v/\partial x_i)+\phi=0$ for given $\phi(x_1, \cdots, x_n)$ in G. Other theorems concern the invariance of dispersiveness under small deformations of the vectors X. *W. Kaplan.*

Adamov, N. V. On certain transformations not changing the integral curves of a differential equation of the first order. Amer. Math. Soc. Translation no. **31**, 58 pp. (1950).

Translated from Mat. Sbornik N.S. **23**(65), 187–228 (1948); these Rev. **10**, 250.

Pinney, Edmund. The nonlinear differential equation $y''+p(x)y+cy^{-3}=0$. Proc. Amer. Math. Soc. **1**, 681 (1950).

The author states, without proof, that the general solution of the differential equation in the title is a simple elementary function of the solutions of the differential equation $y''(x)+p(x)y(x)=0$. *W. Wasow* (Cambridge, Mass.).

Mitrinovitch, D. S., et Vidav, I. Sur un équation différentielle. Bull. Soc. Math. Phys. Macédoine **1**, 21–27 (1950). (Serbo-Croatian and Slovenian. French summary)

Two methods for the integration of $y'^2+y^2=\alpha\sin x$ are given and compared. *W. Feller* (Princeton, N. J.).

Moshinsky, Marcos. On one-dimensional boundary value problems of a discontinuous nature. Bol. Soc. Mat. Mexicana **4**, 1–25 (1947). (Spanish)

Let (a, b) be a finite closed real number interval; c be such that $a<c<b$; $p_1(x)$, $q_1(x)$, $w_1(x)$ real-valued continuous functions on (a, c); $p_2(x)$, $q_2(x)$, $w_2(x)$ continuous functions on (c, b); and, further, $p_1(x)>0$, $p_1'(x)$ is continuous on (a, c), while $p_2(x)>0$, $p_2'(x)$ is continuous on (c, b). Also, one of the four numbers, $p_1(c)-p_2(c)$, $q_1(c)-q_2(c)$, $w_1(c)-w_2(c)$, $p_1'(c)-p_2'(c)$, is not zero. The following one-dimensional eigenvalue problem is considered: to determine all real numbers λ and functions $u(x)\not\equiv 0$, continuous on (a, b) and with continuous second derivatives on (a, c) and (c, b), such that $(p_1u')'+q_1u+\lambda w_1u=0$ on (a, c), $(p_2u')'+q_2u+\lambda w_2u=0$ on (c, b), and further $Au(a)+Bu'(a)=0$, $Cu(a)+Du'(a)=0$, $u(c-0)=u(c+0)$, $p_1(c)u'(c-0)=p_2(c)u'(c+0)$, where A, B, C, D are real constants, such that zero is not an eigenvalue of the problem. The eigenvalue problem is shown to be equivalent to the solution of an integral equation

$$u(x)=\lambda\left[\int_a^c K(x, t)w_1(t)u(t)dt+\int_c^b K(x, t)w_2(t)u(t)dt\right],$$

where the function K is a certain kernel analogous to Green's function. The Hilbert-Schmidt theory of integral equations [Courant and Hilbert, Methoden der mathematischen Physik, vol. I, 2d ed., Berlin, 1931, chapters III and V] is applied to this integral equation. A special case of the above eigenvalue problem arises in finding the displacement of a vibrating string with piecewise constant density. *J. B. Diaz* (College Park, Md.).

Cimmino, Gianfranco. Sui problemi ai limiti per le equazioni differenziali lineari. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **6**, 205–225 (1950).

Let $M_nv=\sum_{j=0}^n(-1)^{(n-j)}(p_jv)^{(j)}$ denote the adjoint of the expression $L_nu=\sum_{j=0}^np_j(x)u^{(j)}$. It is shown that the problem of finding a solution of the nonhomogeneous differential equation $M_nv=g(x)$ assuming preassigned values a_k at the points x_k, $k=1, \cdots, n$, is equivalent to the solution of a certain integral equation. This integral equation differs in two essential respects from the familiar ones based on Green's function: Firstly, its kernel is a relatively simple, explicitly given function of the data, which does not involve the derivatives of the coefficients $p_j(x)$; secondly, adjoint differential equations do not correspond to kernels that are the transpose of each other. Similar integral equations are then derived for problems obtained by replacing the "boundary" conditions $v(x_k)=a_k$ by more general ones. In the second part of the paper it is shown that the equivalent integral equation can be used to define a generalized adjoint problem to every problem of the form $L_nu=f(x)$, $u(x_k)=a_k$, $k=1, \cdots, n$, even when the coefficients $p_j(x)$ are only supposed to be summable and if $f(x)$ is quadratically summable. In terms of this generalization an alternative of the usual type is then shown to be true: If the generalized adjoint homogeneous problem possesses only the trivial solution, then the original problem has a solution. In the opposite case the original problem is solvable, if and only if $f(x)$ and the a_k satisfy certain subsidiary equalities. *W. Wasow* (Cambridge, Mass.).

***Horn, J. Partielle Differentialgleichungen.** 4th ed. Walter de Gruyter & Co., Berlin, 1949. viii+228 pp. 14.00 DM.

Vierte, unveränderte Auflage.

Cramlet, Clyde M. A generalization of a theorem of Jacobi on systems of linear differential equations. Canadian J. Math. **2**, 420–426 (1950).

The theorem of Jacobi in question states that (i) with every equation $a^i\partial u/\partial x^i=0$ there are associated nonzero multipliers M for which $\partial(Ma^i)/\partial x^i=0$ and (ii) the knowledge of such a multiplier and of all but one solution in a complete set leads to an exact equation for a solution completing the set. The present paper points out the existence of multipliers which serve simultaneously for all equations of a complete system $a_j^i\partial u/\partial x^i=0$ and make (ii) true for the system. *J. M. Thomas* (Durham, N. C.).

Landis, E. M. An example of nonuniqueness of solution of Cauchy's problem for a system of the form

$$\frac{\partial u_i}{\partial t}=\sum_j A_{ij}\frac{\partial u_j}{\partial x}+\sum_j B_{ij}u_j+f_i \quad (i, j=1, 2).$$

Mat. Sbornik N.S. **27**(69), 319–323 (1950). (Russian)

Myškis [Doklady Akad. Nauk SSSR (N.S.) **58**, 21–24 (1947); Uspehi Matem. Nauk (N.S.) **3**, no. 2(24), 3–46 (1948); these Rev. **9**, 354; **10**, 302] gave an example of a system

$$\text{(1)} \qquad \begin{aligned}\partial u/\partial t&=a_1\partial u/\partial x+b_1\partial v/\partial x+c_1u+d_1v,\\ \partial v/\partial t&=a_2\partial u/\partial x+b_2\partial v/\partial x+c_2u+d_2v,\end{aligned}$$

with continuously differentiable coefficients, having an infinitely differentiable nontrivial solution $u(t, x)$, $v(t, x)$ defined over the whole (t, x)-plane, having zero Cauchy data on the x-axis: $u(0, x)=v(0, x)=0$. However, u and v vanish on a subset of the (t, x)-plane having the origin $(0, 0)$ as a limit point. The present paper gives another example of a system (1) of the same nature, with a solution $u(t, x)$, $v(t, x)$ which is infinitely differentiable on the square $0\leqq t\leqq 1$, $0\leqq x\leqq 1$, satisfies $u(0, x)=v(0, x)=0$ for $0\leqq x\leqq 1$, but further $u(t, x)\neq 0$, $v(t, x)\neq 0$, for $t\neq 0$. *J. B. Diaz* (College Park, Md.).

Cinquini Cibrario, Maria. Sopra la teoria delle caratteristiche per i sistemi di equazioni quasi-lineari alle derivate parziali del primo ordine. Ann. Scuola Norm. Super. Pisa (3) **3** (1949), 161–197 (1950).

The author gives a discussion of Goursat problems associated with the characteristic manifolds of a system of equations of the form

$$\text{(1)} \qquad \sum_{j=1}^{n}\left(A_{ij}\frac{\partial z_j}{\partial x}+B_{ij}\frac{\partial z_j}{\partial y}\right)=C_i, \qquad i=1, \cdots, n,$$

where A_{ij}, B_{ij}, C_i are given functions of $x, y, z_1, \cdots, z_n$ and $\det(A_{ij})\neq 0$. The characteristic equation of (1) is given by (2) $\det(A_{ij}\rho-B_{ij})=0$. The hyperbolic character of the system corresponds to the assumption that the roots ρ of (2) are real and distinct, or more generally are real, of constant multiplicity, and such that for a root of multiplicity ν the matrix $(A_{ij}\rho-B_{ij})$ has rank $n-\nu$. Along a curve Γ in $(x, y, z_1, \cdots, z_n)$-space, the A_{ij}, B_{ij}, and the roots of (2) become functions of x alone. The curve Γ is called a characteristic curve belonging to a root ρ of (2), if along Γ, $dy=\rho dx$, $\sum_{i,j}h_iA_{ij}dz_j=\sum_i h_iC_i$ for all h_i satisfying $\sum_i(A_{ij}\rho-B_{ij})h_i=0$, $j=1, \cdots, n$. It is proved (under suitable regularity assumptions) that among the infinitely many integral surfaces passing through a characteristic curve Γ belonging to a simple root ρ there is precisely one containing a curve with a given projection in (x, y, z_1)-space. There is also precisely one integral surface through Γ containing a second characteristic curve belonging to a different ρ, and having a prescribed projection in (y, z_1)-space. In general there exists no integral surface through two given intersecting characteristic curves, except in the case where the number of distinct roots of (2) reduces to 2. This paper also contains a discussion of the Cauchy problem for (1), and of the characteristic strips of various orders associated with a characteristic curve. *F. John.*

Courant, R., and Lax, P. Method of characteristics for the solution of nonlinear partial differential equations. Symposium on theoretical compressible flow, 28 June 1949. Naval Ordnance Laboratory, White Oak, Md., Rep. **NOLR-1132**, pp. 61–71 (1950).

Des nombreux problèmes de mécanique des fluides conduisent à des systèmes quasi-linéaires d'équations aux dérivées partielles à plusieurs fonctions inconnues de deux variables indépendantes. Le classification des systèmes semi-linéaires (non dégénérés) se fait très simplement grâce à la théorie classique des "diviseurs élémentaires": Les matrices composantes sont dites suivant les cas hyperboliques, elliptiques, hyperbolico-paraboliques d'ordre k, elliptico-paraboliques d'ordre k.

Pour les systèmes totalement hyperboliques, semi-linéaires, le problème de Cauchy se ramène à un système d'équations intégrales (du type Volterra), qui se résout aisément par itération. L'étude des systèmes quasi-linéaires se heurte à plusieurs difficultés; elles ont pu être surmontées par des methodes variées, en particulier: approximation par des systèmes semi-linéaires [Friedrichs, Amer. J. Math. **70**, 555–589 (1948); ces Rev. **10**, 41; Douglis, Thesis, New York University, 1949]; approximation par des systèmes quasi-linéaires simples, de solutions connues par ailleurs [Schauder, Comment. Math. Helv. **9**, 263–283 (1937); Lax, Thesis, New York University, 1949; Douglis, loc. cit.]. Un problème mixte, assez général, a été résolu par Göldner [Thesis, New York University, 1949]. Les autres cas ont été aussi examinés, mais semblement beaucoup moins bien connus pour l'instant. En revanche, pour le cas totalement hyperbolique, on est en possession de méthodes bien adaptées au calcul numérique. *M. Janet* (Paris).

Bouligand, Georges. Sur certaines équations

$$f(x, y, z, p, q)=0.$$

Gaz. Mat., Lisboa **11**, no. 43, 1–6 (1950).

This paper raises certain questions about Cauchy's problem for the partial differential equation $f(x, y, z, p, q)=0$, where f is a polynomial quadratic in p, q, and deals mainly in generalities concerning these questions. It is remarked that two such equations do not in general have two one-parameter families of orthogonal integral surfaces.

J. M. Thomas (Durham, N. C.).

Lehto, Olli. On Hilbert spaces with a kernel function. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **74**, 12 pp. (1950).

The author applies a criterion of Aronszajn [Proc. Cambridge Philos. Soc. **39**, 133–153 (1943); these Rev. **5**, 38] for the existence of a reproducing kernel in a given Hilbert space to two particular cases. The main result, referring to classes of solutions of partial differential equations of elliptic type, was obtained previously by Bergman and Schiffer [Duke Math. J. **14**, 609–638 (1947); these Rev. **9**, 187], a fact the author fails to point out although the literature in question is quoted. *Z. Nehari* (St. Louis, Mo.).

Titchmarsh, E. C. Some theorems on perturbation theory. II. Proc. Roy. Soc. London. Ser. A. **201**, 473–479 (1950).

In a previous paper [same Proc. Ser. A. **200**, 34–46 (1949); these Rev. **11**, 596] the author considered the partial differential equation $\nabla^2\varphi+\{\lambda-q(x, y)-\epsilon\sigma(x, y)\}\varphi=0$, in which ϵ is a small positive parameter. The equation with $\epsilon>0$ is to be regarded as a perturbation of that with $\epsilon=0$. It was assumed there that each λ eigenvalue had just one eigenfunction corresponding to it. The present paper extends the discussion to the case in which the smallest eigenvalue when $\epsilon=0$ is multiple, with k normal orthogonal eigenfunctions $\psi_n(x, y)$ corresponding to it. The region is the whole plane. The spectra, for both $\epsilon=0$ and $\epsilon>0$, are assumed to be discrete, with ∞ as the only limit point. It is assumed that $\sigma(x, y)\geqq 0$, that $\int_0^\infty\int_0^{2\pi}(\partial\psi_n/\partial r)^2 r dr d\theta$ is

convergent for each n, and that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma^2\psi_n^2dx\,dy$ is finite for every n. *R. E. Langer* (Madison, Wis.).

Jacobson, A. W. The Green's functions for the rectangle obtained by the finite Fourier transformations. Proc. Amer. Math. Soc. 1, 682–686 (1950).

The author investigates the two-dimensional steady flow of heat in a rectangle by means of the finite Fourier transformation. He expresses the Green function of this problem in terms of the inverse cosine transform $B_{12}(x, u)$ of $(n \sinh ny_0)^{-1} \cosh nu$. The expression of B_{12} in terms of Weierstrass' sigma function proves the identity of the author's Green function with that already known for this problem. [Cf. also Bull. Amer. Math. Soc. **55**, 804–809 (1949); Quart. Appl. Math. 7, 293–302 (1949); these Rev. 11, 100.] *A. Erdélyi* (Pasadena, Calif.).

Ladyženskaya, O. On the solution of mixed problems for hyperbolic equations. Doklady Akad. Nauk SSSR (N.S.) **73**, 647–650 (1950). (Russian)

Let n be a positive integer, and Ω be a bounded open set in $X=(x_1, \cdots, x_n)$ space. The author announces (under certain smoothness and growth restrictions on the coefficients of the equation) the following result: There exists one and only one real-valued function u with continuous second derivatives in Ω, satisfying the conditions

$$\frac{\partial^2 u}{\partial t^2}=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(X)\frac{\partial u}{\partial x_j}\right)+\sum_{i=1}^{n}b_i(X)\frac{\partial u}{\partial x_i}+c(X)u-\varphi(X,t),$$

for $t\geqq 0$ and $X\epsilon\Omega$; $u(X, 0)=0$, $\partial u/\partial t(X, 0)=0$, for $X\epsilon\Omega$; for each $t\geqq 0$, u has the boundary value zero in the mean sense, i.e., $\lim_{r\to 0} r^{-1}\int_{S_r}u^2d\Omega=0$, where S_r is a "boundary strip of width r"; and also $\int_\Omega(u^2+u_t^2+\sum_{i=1}^{n}u^2_{x_i})\,d\Omega$ exists for each $t\geqq 0$ and is bounded above on each finite interval $0\leqq t\leqq T$. The construction of the function u is carried out by first performing a Laplace transformation

$$v(X,\lambda)=\int_0^\infty u(X,t)e^{-\lambda t}dt,$$

and then dealing with the resulting equation for v by replacing it by a finite difference equation, using the methods of Courant, Friedrichs, and Lewy [Math. Ann. **100**, 32–74 (1928)], and Sobolev [Rec. Math. [Mat. Sbornik] N.S. **2**(44), 465–499 (1937)]. It is stated that when $n=2$ the solution $u(X, t)$ actually vanishes on the boundary. *J. B. Diaz* (College Park, Md.).

Robinson, A. On the integration of hyperbolic differential equations. J. London Math. Soc. **25**, 209–217 (1950).

The author solves Cauchy's problem for a linear hyperbolic equation

$$(1)\qquad \sum_{k=0}^{n}\sum_{l=0}^{k}\alpha_{kl}\frac{\partial^k z}{\partial x^{k-l}\partial y^l}=\alpha_0,\qquad (\alpha_{n0}\neq 0).$$

Cauchy data for the solution $z(x, y)$ are prescribed on $x=0$. The characteristic equation $\sum_{i=0}^{n}(-1)^i\alpha_{ni}\gamma^{n-i}=0$ is assumed to have distinct real roots $\gamma_k=\gamma_k(x, y)$. The problem is reduced to the solution of a Cauchy problem for a system of first order quasi-linear equations, which is of the form (2) $\partial f_i/\partial x+c_i\partial f_i/\partial y=F_i(x, y, f_1, \cdots, f_m)$, $i=1, \cdots, m$. The corresponding system of integral equations obtained by integrating (2) along the curve $dy=c_idx$ is solved by iteration. The author's method of solution is closely related to the solution of the Cauchy problem for nonlinear hyperbolic systems of first order equations given by Courant and Lax [Comm. Pure Appl. Math. **2**, 255–273 (1949); these Rev. 11, 441]. *F. John* (Los Angeles, Calif.).

★**Bureau, Florent. Quelques questions de géométrie suggérées par la théorie des équations aux dérivées partielles totalement hyperboliques.** Colloque de géométrie algébrique, Liège, 1949, pp. 155–176. Georges Thone, Liège; Masson et Cie., Paris, 1950. 200 Belgian francs; 1400 French francs.

The author draws attention to a number of questions suggested by his work on the Cauchy problem for hyperbolic differential equations of arbitrary order and in any number of independent variables. Among these questions are the problems of describing the real topological structure of an algebraic surface and the study of the integrals over algebraic manifolds encountered in evaluating the expressions for the elementary and auxiliary solutions. *F. John* (Los Angeles, Calif.).

Salehov, G. S. On the problem of Cauchy-Kovalevskaya for a class of linear partial differential equations in the domain of arbitrarily smooth functions. Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 355–366 (1950). (Russian)

Consider the linear partial differential equation

$$(*)\qquad \partial^p z/\partial t^p-\epsilon t^m Dz=0,$$

where $\epsilon=\pm 1$; m is a nonnegative integer,

$$D=\sum_{(\lambda)}A_{\lambda_1\cdots\lambda_s}\partial^\lambda/\partial x_1^{\lambda_1}\cdots\partial x_s^{\lambda_s},$$

and $A_{\lambda_1\cdots\lambda_s}$ are constant coefficients. The author raises the following two questions concerning the solution of the Cauchy-Kowalewsky problem for (*): (1) What necessary and sufficient conditions must be satisfied by the given initial data $\varphi_k(x_1, \cdots, x_s)$, defined in a certain closed domain g of $(x_1, \cdots, x_s)$-space, in order that the Cauchy problem for (*) with initial conditions

$$(**)\qquad \partial^k z(0, x_1, \cdots, x_s)/\partial t^k=\varphi_k(x_1, \cdots, x_s),$$

$k=0, 1, \cdots, p-1$, possess a solution which is analytic in the (real or complex) variable t in a neighborhood of $t=0$? (2) If such a solution exists, what is its analytic behavior with respect to $(x_1, \cdots, x_s)$? In an earlier paper [Doklady Akad. Nauk SSSR (N.S.) **59**, 857–859 (1948); these Rev. 9, 441] the author considered these questions for the special case of (*): $\partial^p z/\partial t^p-\epsilon t^m\partial^q z/\partial x^q=0$. A function $\varphi(x_1, \cdots, x_s)$ is said to be of class $\alpha>0$ in a closed domain g with respect to the operator D provided that there exist positive constants M and H such that $|D^n\varphi(x_1, \cdots, x_s)|\leqq M(n!)^\alpha/H^n$, for any $(x_1, \cdots, x_s)$ in g and any $n=1, 2, \cdots$. [This notion is related to that of the classes of infinitely differentiable functions of M. Gevrey, Ann. Sci. École Norm. Sup. (3) **35**, 129–190 (1918).] By a solution of (*) in a domain G is meant a function which satisfies (*) in G and is such that all of its derivatives which appear in D are continuous in G. Question (1) is answered by theorem 1: The Cauchy-Kowalewsky problem (*), (**) has a solution which is analytic with respect to t if and only if the initial data appearing in (**) are of class $\alpha\leqq p$ with respect to D in g. If $\alpha=p$, then the solution is analytic in $|t|<R$, where R is a certain positive number. If $\alpha<p$, then the solution is an entire function with respect to t. Question 2 is answered by theorem 2: If the initial data satisfy the necessary and sufficient conditions of theorem 1, and G is the domain of definition of the solution in $(t, x_1, \cdots, x_s)$-space, then the

solution is (in an obvious sense) of class α with respect to D in any closed, bounded subset of G. *J. B. Diaz.*

Majer, J. Das reine Randwertproblem des ebenen elastischen Keiles. Österreich. Ing.-Arch. 4, 290–303 (1950).

The author derives and discusses some of the properties of a Mellin-transform solution of the boundary value problem of the two-dimensional, biharmonic equation in the wedge-shaped region $0<r<\infty$, $\theta_1<\theta<\theta_2$.

E. Reissner (Cambridge, Mass.).

Integral Equations

Germay, R. H. Sur des équations intégrales récurrentes. I. Bull. Soc. Roy. Sci. Liège 19, 142–149 (1950).

Germay, R. H. Sur des équations intégrales récurrentes. II. Bull. Soc. Roy. Sci. Liège 19, 198–203 (1950).

The author considers the particular system of Volterra equations,

$$\varphi_i(x)=f_i(x)+\int_a^x\{K_{i,i}(x,s)\varphi_i(s)+K_{i,i+1}(x,s)\varphi_{i+1}(s)\}\,ds,$$

$i=1,\cdots,n$. He shows that if f_i and K_{ij} are continuous there exists a unique continuous solution $\varphi_i(x)$. By the use of Dirichlet's formula he is able to write the approximating functions $\varphi_{n,\mu}(x;\lambda)$ as polynomials of the μth degree in λ where the coefficients involve integrals of the iterated kernels of the K_{ij}. The existence and uniqueness proofs are carried out in the usual manner. *I. A. Barnett* (Cincinnati, Ohio).

Heinhold, J. Einige mittels Laplace-Transformation lösbare Integralgleichungen. I. Math. Z. 52, 779–790 (1950).

Several integral equations of Volterra type whose kernels involve Bessel functions are solved here by using Laplace transformations. The equation

$$Y(t)-\alpha^2t\int_0^t H(\alpha(t^2-\tau^2)^{\frac{1}{2}})\,Y(\tau)\,d\tau=G(t),$$

where $H(x)=J_1(x)/x$ and α is a complex constant, is an example. The author first proves that the Laplace transform of the left-hand member of this integral equation is $sy(p)/p$, where $p=(s^2+\alpha^2)^{\frac{1}{2}}$ and $y(s)$ is the transform of the function $Y(t)$. He solves such integral equations rigorously, usually in terms of integrals involving Bessel functions and the prescribed function $G(t)$. The types of equations that can be solved by this method are associated with operational properties of transforms. *R. V. Churchill.*

Krein, M. G. On the Sturm-Liouville boundary problem in the interval $(0,\infty)$ and on a class of integral equations. Doklady Akad. Nauk SSSR (N.S.) 73, 1125–1128 (1950). (Russian)

Starting from two real functions $\psi(x)$, $\chi(x)$ and a bounded nondecreasing distribution function $\sigma(x)$, the author sets up the integral equation $\varphi(x)=\lambda\int_0^\infty K(x,s)\varphi(s)\,d\sigma(s)$, where $K(x,s)=\psi(x)\chi(s)$ for $x\leqq s$, $K(x,s)=\psi(s)\chi(x)$ for $x\geqq s$. He also defines $\psi(x;\lambda)$, $\chi(x;\lambda)$ by

$$\psi(x;\lambda)=\psi(x)+\lambda\int_0^x V(x,s)\psi(s;\lambda)\,d\sigma(s),$$

and similarly for $\chi(x;\lambda)$, where $V(x,s)=\psi(s)\chi(x)-\psi(x)\chi(s)$, and further

$$D_0(\lambda)=1-\lambda\int_0^\infty \psi(s;\lambda)\chi(s)d\sigma(s),$$

$$D_1(\lambda)=-\lambda\int_0^\infty \psi(s;\lambda)\psi(s)d\sigma(s),$$

$$E_0(\lambda)=\lambda\int_0^\infty \chi(s;\lambda)\chi(s)d\sigma(s),$$

$$E_1(\lambda)=1+\lambda\int_0^\infty \chi(s;\lambda)\psi(s)d\sigma(s),$$

where $D_0(\lambda)$ is the Fredholm determinant of the first integral equation. Theorem 1 then asserts that for real α, the function $F_\alpha(\lambda)=(\cos\alpha E_0(\lambda)+\sin\alpha E_1(\lambda))/(\cos\alpha D_0(\lambda)+\sin\alpha D_1(\lambda))$ maps the upper half λ-plane into a part thereof. From theorems of Čebotarev [Čebotarev and Meĭman, The Routh-Hurwitz problem for polynomials and entire functions . . . , Trudy Mat. Inst. Steklov. 26 (1949), chapters IV, V; these Rev. 11, 509] it is said to follow that the zeros of the denominator and numerator are real, simple, and separate one another. A partial fraction expansion of $F_\alpha(\lambda)$ is also given. Theorem 2 states that $D_0(\lambda)$, $D_1(\lambda)$, $E_0(\lambda)$, and $E_1(\lambda)$ belong to the class (N) of integral functions $f(z)$ for which $\lim_{|z|\to\infty}|z|^{-1}\log|f(z)|<\infty$, and

$$\int_{-\infty}^\infty(1+\lambda^2)^{-1}|\log|f(\lambda)||\,d\lambda<\infty.$$

Theorem 3 deals with the zeros of $D_0(\lambda)$, also the eigenvalues of the first integral equation. It is stated that if $\sum_{\lambda_j<0}|\lambda_j|^{-\frac{1}{2}}<\infty$, then $D_0(\lambda^2)$ belongs to the class (N) and the limit $\lim_{n\to\infty}n^2\lambda_n^{-1}$ exists [see the author, Bull. Acad. Sci. URSS. Sér. Math. [Izvestiya Akad. Nauk SSSR] 11, 309–326 (1947), theorem 5, the proof of which is said to need correction; these Rev. 9, 179]. Theorem 4 gives an application to the Sturm-Liouville problem of

$$\varphi''+q(x)\varphi+\lambda\rho(x)\varphi=0$$

over $(0,\infty)$, in the "degenerate" case in which for every complex λ there are two solutions for which $\int_0^\infty\varphi^2\rho dx<\infty$. If $\int_0^\infty\rho dx=\infty$ then certain boundary problems have negative eigenvalues $\lambda_{-1},\lambda_{-2},\cdots$ for which $\sum_{n=1}^\infty|\lambda_{-n}|^{-\frac{1}{2}}=\infty$.

F. V. Atkinson (Ibadan).

Fréchet, Maurice. Sur certaines équations intégrales que l'on rencontre dans les applications. Bull. Soc. Math. Belgique 2 (1948–1949), 33–35 (1950).

This note points out that the ultimate behavior of iterated kernels posed by "probabilités en chaîne," viz., that of $K^{(n)}(M,P)$ as $n\to\infty$ with $K(M,P)\geqq 0$, and $\int_V K(M,P)dP=1$ can be deduced from considerations by the author in previous papers [Quart. J. Math., Oxford Ser. (1) 5, 106–144 (1934); J. Math. Pures Appl. (9) 15, 251–270 (1936)].

T. H. Hildebrandt (Ann Arbor, Mich.).

Lehmann, N. J. Berechnung von Eigenwertschranken bei linearen Problemen. Arch. Math. 2, 139–147 (1950).

The eigenvalue problem considered is the linear integral equation (*) $y(x)=\lambda\int K(x,s)y(s)\,ds$, where the kernel K is real, symmetric, square integrable, and continuous in the mean, and the integral is taken over a bounded finite-dimensional domain. The paper is an account of the main results, without proofs, of the author's thesis [Z. Angew. Math. Mech. 29, 341–356 (1949); 30, 1–16 (1950); these Rev. 11, 599]. The chief purpose is to present a method for

finding lower and upper bounds for the eigenvalues of (*), which are known to be real. *J. B. Dias.*

Vaĭnberg, M. M. Existence theorems for the characteristic values of a class of systems of nonlinear integral equations. Mat. Sbornik N.S. **26(68)**, 365–394 (1950). (Russian)

The author studies the system

$$(1)\qquad \mu_i u_i(x)=\int_B K_i(x,y)g_i(u_1(y),\cdots,u_n(y),y)\,dy,$$

$i=1,\cdots,n$; the kernels $K_i(x,y)$ are symmetric and positive; the $g_i(u,x)$ are continuous in $u=(u_1,\cdots,u_n)$ (u_j real) for x fixed in the bounded domain (in m-space) B, are measurable in x (in B) for u fixed, and $g_i(u,x)=\partial G(u,x)/\partial u_i$, $g(0,x)\equiv G(0,x)\equiv 0$. The function

$$\psi(x)=(\psi_1(x),\cdots,\psi_n(x))\in L_{2,n}(B)$$

and of nonzero norm is termed a c.f. (characteristic function) of (1) if ψ satisfies (1) for some real values of the μ_i; $\mu^{(0)}=(\mu_1^{(0)},\cdots,\mu_n^{(0)})$, for which (1) has a c.f., is termed a c.v. (characteristic value). Under various additional hypotheses on K_i and g_i, it is shown that there exists at least a denumerable infinity of c.f.'s of (1) in $L_{2,n}(B)$. The main theorem asserts that, if the aggregate of functions $g_i(u,x)$ defines a continuous operator hu in $L_{2,n}(B)$ and

$$0<\int\int K_i^2(x,y)\,dx\,dy<\infty,$$

then (1) has at least a denumerable infinity of c.f.'s in $L_{2,n}(B)$, tending to zero according to norm. Conditions for continuity of the operator hu are examined in some detail. The case of bounded kernels is also considered.

W. J. Trjitzinsky (Urbana, Ill.).

Musina, S. S. Approximate solution of a class of nonlinear integral equations. Mat. Sbornik N.S. **27(69)**, 171–174 (1950). (Russian)

In this note the equation

$$R(u)\equiv u(x)-\lambda\int_a^b K(x,y,u(y))\,dy=0$$

is considered and it is assumed that

$$|K(x,y,u_1)-K(x,y,u_2)|\leqq C|u_1-u_2|$$

and $K>0$, $\partial K/\partial u>0$, $\lambda>0$, for $a\leqq x,y\leqq b$, $-L\leqq u,u_1,u_2\leqq L$. Then if λ is sufficiently small and $R(\phi_0)>0$, $\phi_n=R(\phi_{n-1})$, $R(\psi_0)<0$, $\psi_n=R(\psi_{n-1})$, the most elementary calculation shows that the sequence $\{\phi_n(x)\}$ converges decreasingly and the sequence $\{\psi_n(x)\}$ converges increasingly to the unique solution of $R(u)=0$. *M. Golomb* (Lafayette, Ind.).

Lewis, H. W. Multiple scattering in an infinite medium. Physical Rev. (2) **78**, 526–529 (1950).

If the direction of motion of a charged particle is given by a unit vector $\mathbf{v}$, the cross section for scattering per unit solid angle by σ, the position by $\mathbf{r}$, the equation governing the distribution function $f(\mathbf{r},\mathbf{v},s)$, where s is the arc length traversed by the particle, is

$$\frac{\partial f}{\partial s}+\mathbf{v}\cdot\operatorname{grad} f=N\int[f(\mathbf{r},\mathbf{v}',s)-f(\mathbf{r},\mathbf{v},s)]\sigma(|\mathbf{v}-\mathbf{v}'|)\,d\mathbf{v}',$$

where N is the number of particles per unit volume and the integration on the right-hand side is over the entire solid angle. The solution of this equation is sought which satisfies the boundary condition $f(\mathbf{r},\mathbf{v},0)=\delta(\mathbf{r})\delta(\mathbf{v})$ (δ is Dirac's δ-function) corresponding to a single particle incident at the origin and moving in the z-direction and which is bounded as $|\mathbf{r}|\to\infty$. Expanding f in normalized surface harmonics $Y_{lm}(\mathbf{v})$ in the form $f=\sum_{lm}f_{lm}(\mathbf{r},s)Y_{lm}(\mathbf{v})$ we obtain the equation (1) $\partial f_{lm}/\partial s+\kappa_l f_{lm}=-\sum_{\lambda\mu}\operatorname{grad} f_{\lambda\mu}\cdot\mathbf{Q}_{lm,\lambda\mu}$, where $\mathbf{Q}_{lm,\lambda\mu}=\int Y^*_{lm}\mathbf{v}Y_{\lambda\mu}d\mathbf{v}$ is a constant vector which is zero if $|\lambda-l|$ or $|\mu-m|$ is greater than unity and

$$\kappa_l=2\pi N\int_0^\pi \sigma(\vartheta)[1-P_l(\cos\vartheta)]\sin\vartheta\,d\vartheta.$$

The corresponding boundary conditions for equation (1) are $f_{lm}(\mathbf{r},0)=[(2l+1)/4\pi]^{\frac{1}{2}}\delta_{m0}\delta(\mathbf{r})$. The solution for the angular distribution $F(\mathbf{r},s)=\int f(\mathbf{r},\mathbf{v},s)\,d\mathbf{r}$ is then obtained in the form

$$F(\mathbf{r},s)=(1/4\pi)\sum_{l=0}^{\infty}(2l+1)P_l(\cos\vartheta)k_l(s);$$

$$k_l(s)=\exp\left(-\int_0^s\kappa_l ds,\right)$$

where κ_l, now regarded as a function of s, allows for the energy dependence of the scattering cross section.

The case $\sigma(\vartheta)=c/(1-\cos\vartheta+2\beta)^2$, where $\beta=\text{constant}\ll 1$ and c is a constant, is considered in detail. For this case the author shows that

$$\kappa_l=C\left[-1+\sum_{\nu=0}^{l-1}\frac{\beta^\nu}{\nu!(\nu+1)!}\frac{(l+\nu+1)!}{(l-\nu-1)!}G_{l\nu}\right],$$

where C is a constant and

$$(*)\qquad G_{l\nu}=\left[\log(1/\beta)+\left(\sum_1^{\nu}+\sum_1^{\nu+1}-\sum_1^{l-\nu-1}-\sum_1^{l+\nu+1}\right)m^{-1}\right].$$

(In (*) it is to be understood that sums in which the upper limit is less than unity are to be omitted.) Considering the quantities

$$H_{lm}^{(n)}(s)=\int_{-\infty}^{+\infty}z^n f_{lm}(\mathbf{r},s)\,d\mathbf{r}\text{ and }h_{lm}^{(n)}(s)=\int_{-\infty}^{+\infty}x^n f_{lm}(\mathbf{r},s)\,d\mathbf{r}$$

(which are the moments of the longitudinal and the transverse distributions), the author shows that they satisfy a system of linear differential equations with constant coefficients which can be solved successively.

S. Chandrasekhar (Williams Bay, Wis.).

Functional Analysis, Ergodic Theory

★Kantorovič, L. V., Vulih, B. Z., and Pinsker, A. G. Funkcional'nyĭ analiz v poluuporyadočennyh prostranstvah. [Functional Analysis in Partially Ordered Spaces]. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 548 pp.

The volume under review contains an encyclopedic discussion of the theory of partially ordered linear spaces, as developed by Freudenthal [Akad. Wetensch. Amsterdam, Proc. **39**, 641–651 (1936)], Kantorovič [e.g., C. R. (Doklady) Acad. Sci. URSS (N.S.) **12** (**1936 III**), 9–14; Rec. Math. [Mat. Sbornik] N.S. **2(44)**, 121–165 (1937)], and many other writers. [For a basic list of original memoirs, cf. G. Birkhoff, Lattice Theory, Amer. Math. Soc. Colloquium Publ., v. 25, 2d ed., New York, 1948; these Rev. **10**, 673.] The work is divided into an introduction, thirteen chapters, and two appendices. In the introduction there is an extended historical sketch in which the work of non-Soviet mathe-

maticians in the field of functional analysis is covered in seven lines. The scientific portion of the book has as its basic object the study of complete vector lattices. Such spaces are here referred to as K-spaces [nomenclature introduced by Pinsker according to the authors]. The authors change their axioms whenever convenient, and succeed in defining, in the course of the exposition, no fewer than twenty different types of vector lattices. Chapter I deals with the elementary properties of K-spaces. Great numbers of identities involving "$\vee$," "$\wedge$," "$+$," "$-$," and "$|\cdot|$" are proved; o-convergence and $*$-convergence, here referred to as t-convergence, are defined and discussed, and a few standard examples of K-spaces are produced. In chapter II, certain special subspaces of K-spaces, called components, are introduced. A linear subspace E of a K-space X is called a component if $x\epsilon E$, $y\epsilon X$, and $|x|\geqq|y|$ imply $y\epsilon E$, and if E contains the suprema in X of all subsets of E which are bounded above in X. It is shown that every K-space X is in a certain sense a subdirect sum of pair-wise lattice disjoint components. Chapter III presents Freudenthal's results [loc. cit.] on integral representation of elements of K-spaces, here considerably extended. A theorem of G. Birkhoff [Proc. Nat. Acad. Sci. U. S. A. **24**, 154–159 (1938)], to the effect that the set of all components of a K-space is a complete Boolean algebra, is also proved. In chapter IV, we find results, in part new, concerning extensions of K-spaces. It is first shown that any complete Boolean algebra can serve as the Boolean algebra of components of a K-space. This is done by a construction introduced by Kakutani [Ann. of Math. (2) **42**, 523–537 (1941); these Rev. **2**, 318]. Next, it is shown that every K-space can be imbedded in another K-space having a certain strong completeness property. It is also shown that every such complete K-space admits a natural definition of multiplication for every pair of elements, making it a commutative and associative algebra over the real field. Chapters V and VI treat K-spaces in which various additional axioms are supposed satisfied, such as the existence of a metric, a norm, etc. Chapter VII deals with additive homogeneous operators carrying a given K-space into another. Four types of continuity are defined and studied in great detail; and as one would expect, the set of all operators continuous in a certain sense is shown to be a K-space. Chapter VIII presents representation theorems for various additive, homogeneous, and continuous operators. The usual integral representations for linear functionals on spaces of continuous real functions, L_p, l_p, L_∞, etc., are shown to have their abstract analogues. A large number of special examples are worked out. Chapter IX begins with an obvious analogue of the Hahn-Banach theorem for mappings of any linear space into a K-space, and proceeds to a long collection of theorems dealing with extensibility of operators of various kinds, with the preservation of various properties. The Daniell construction of the Lebesgue integral (for the case of functions vanishing outside of bounded sets) from the Riemann integral for continuous functions appears as a very special case of one of these extension theorems. A noteworthy fact is that Lebesgue's theorems on term-by-term integration are already contained in the general theorem employed. The classical moment problems for the line and for bounded intervals are also solved by application of the same general theorem. Chapters X, XI, and XII deal with convergence of sequences of additive, homogeneous, and continuous operators, with operators continuous in a certain strong sense, and with application of these results to integral and differential equations. Here the principal tools are a number of fixed point theorems for abstract operators. Chapter XIII presents a survey of the known results concerning the concrete representation of K-spaces, as well as some new facts concerning such representations. For example, it is shown that every K-space can be represented as a linear subspace of the space of (possibly infinite-valued) real continuous functions on a certain extremally disconnected compact Hausdorff space. The linear subspace in question contains with x all y such that $|y|\leqq|x|$. It is also shown that K-spaces of other types are really spaces L_1 for suitable measure spaces [see Kakutani, loc. cit.].

Reviewer's remarks: This book seems to suffer from a number of shortcomings. First, as to omissions. Nowhere is there given a definition of the weak topology, about which assuredly some interesting facts could be established. Moore-Smith limits in K-spaces, which are surely a natural device, have been ignored. Furthermore, the concrete examples presented are all of a perfectly standard kind, providing little real illumination for the general theory expounded. Second, the proliferation of definitions and axiom systems is so great that concentrated attention is required to keep track of what has been proved and under what restrictions. Third, the applications to analysis, which are stated to be the great achievement of the theory, seem for the most part quite standard. Most of the proofs on extensibility and fixed points appear to be obtained by restating known proofs from classical analysis in a form amenable to K-spaces. It appears that no important new analytic facts have been obtained. The typography is excellent, and only a few trivial misprints were detected by the reviewer. *E. Hewitt* (Seattle, Wash.).

Širohov, M. F. Functions of elements of partially ordered spaces. Doklady Akad. Nauk SSSR (N.S.) **74**, 1057–1060 (1950). (Russian)

In chapter IV of the treatise reviewed above, an elaborate construction is presented which enables one, in certain cases, to define continuous functions of elements of a K-space. That is, if $F(t_1, \cdots, t_n)$ is a real-valued function defined on n-dimensional Euclidean space, satisfying certain restrictions, then one can define a function $F(x_1, \cdots, x_n)$ with domain contained in $X\otimes\cdots\otimes X_{(n)}$ and range contained in X, for an arbitrary K-space X, which will in a certain sense reflect the properties of the original function F. The present note presents a simplified definition of the induced function $F(x_1, \cdots, x_n)$; very little is asserted or proved about it, however. *E. Hewitt* (Seattle, Wash.).

Krein, M. G., and Rutman, M. A. Linear operators leaving invariant a cone in a Banach space. Amer. Math. Soc. Translation no. **26**, 128 pp. (1950).

Translated from Uspehi Matem. Nauk (N.S.) **3**, no. 1(23), 3–95 (1948); these Rev. **10**, 256.

Eberlein, W. F. Banach-Hausdorff limits. Proc. Amer. Math. Soc. **1**, 662–665 (1950).

A functional $L(x)$ on the Banach lattice m of bounded real sequences $x=(x_0, x_1, \cdots)$ is called a Banach limit if $L(ax+by)=aL(x)+bL(y)$, $L(1)=1$, $L(x)\geqq 0$ for $x\geqq 0$ and $L((x_1, x_2, \cdots))=L((x_0, x_1, \cdots))$. The reviewer has described [Acta Math. **80**, 167–190 (1948); these Rev. **10**, 367] those $x\epsilon m$ for which all the Banach limits coincide. The author calls $L(x)$ a Banach-Hausdorff limit if $L(Hx)=L(x)$ for any regular Hausdorff transformation H. The existence of such

limits is proved [this result is closely related to a theorem of Kreĭn and Rutman, see the preceding review]. All Banach-Hausdorff limits coincide on an $x\epsilon m$ if and only if $-P(-x)=P(x)$, $P(x)$ being the infimum of lim sup y_n, $(y_0, y_1, \cdots)=H(x_0, x_1, \cdots)$ for all regular Hausdorff transformations H with an increasing generating function.

G. Lorentz (Toronto, Ont.).

Nikodým, Otton Martin. Sur les fonctionnelles linéaires. Pseudo-topologie. C. R. Acad. Sci. Paris **229**, 16–18 (1949).

Given a nonempty set S, a pseudo-topology (p.t.) on S is a distinguished class (G) of subsets of S containing S and the empty set ϕ and closed under finite intersections and countable unions. The sets of G are then the pseudo-open (p.o.) sets, their complements the pseudo-closed sets. A real-valued function $f(s)$ on S is termed upper (lower) pseudo semi-continuous if for every real λ the Lebesgue set $[s|f(s)>\lambda]$ ($[s|f(s)<\lambda]$) is p.o. Various classical theorems on continuous and semi-continuous functions involving only a countable number of Lebesgue sets are valid in a p.t. A p.o. set E is called active if there exists an open set α of real numbers and a pseudo-continuous (p.c.) $f(s)$ such that $E=f^{-1}(\alpha)$. Two characterizations of active sets E are: (1) There exists a p.c. $f(s)$ positive on E and null on $S-E$; (2) the characteristic function of E is the limit of a nondecreasing sequence of p.c. functions. The active sets of G constitute a p.t. (G^*) called the reduced p.t. of G. Two p.t. G_1 and G_2 yield the same p.t. reduced when their families of continuous functions coincide. [Reviewer's note. The notion of a p.t. reduced is analogous to complete regularity.]

W. F. Eberlein (Madison, Wis.).

Nikodým, Otton Martin. Sur les fonctionnelles linéaires. Classe régulière de fonctions. Intégration. C. R. Acad. Sci. Paris **229**, 169–171 (1949).

The author considers applications of the notion of a pseudo-topology [cf. the preceding review] to linear functionals. A regular class of functions on the fundamental set S is a linear class of real-valued functions $f(S)$ on S containing the constant functions, absolute values of elements, and common limits of nonincreasing and nondecreasing sequences. The classes of p.c. functions $(C)_G$ and the bounded p.c. functions $(C)_{Gb}$ defined by a p.t. G are regular classes. If (C) is a regular class of functions on S the family G of sets $f^{-1}(\alpha)$ ($f\epsilon(C)$, α open) is a p.t. reduced, and the bounded p.c. functions relative to (G) coincide with the bounded functions of (C). Given a Boolean algebra A of subsets of S with S as unit and a bounded finitely additive set function μ (Jordan measure) defined on A, the integral $\int f d\mu$ of a bounded function measurable (A) is defined by finite vertical partitions; the integral is extended to unbounded functions by the de La Vallée Poussin method. If A is the Boolean extension $B(G)$ of a p.t. (G), $U(f)=\int f d\mu$ is a continuous linear functional on the space of μ-summable and p.c. functions $f(s)$ relative to G topologized by uniform convergence.

W. F. Eberlein (Madison, Wis.).

Nikodým, Otton Martin. Sur les fonctionnelles linéaires. Représentation par des intégrales. C. R. Acad. Sci. Paris **229**, 288–289 (1949).

[Cf. the preceding review.] If U is a continuous linear functional defined on a $(C)_{Gb}$ topologized by uniform convergence, U admits a representation $U(f)=\int f d\mu$, where μ is a Jordan measure defined on the Borel extension $(T)_G$ of (G), and μ is unique if and only if (G) is a Boolean σ-algebra. If (G) is a reduced p.t., a Jordan measure μ on $(B)_G$ has a countably additive extension if and only if $\lim_n \int f_n d\mu=\int(\lim_n f_n) d\mu$ for every bounded sequence f_n of p.c. functions such that $g(s)=\lim_n f_n(s)$ exists and is measurable $B(G)$. Given a linear class (V) of real-valued functions on S and a Jordan measure μ such that every f of (V) is μ-summable, there exists a p.t. (G) such that $(V)\subset(C)_G$ and a Jordan measure ν on $(T)_G$ such that $\int f d\mu=\int f d\nu$ for every f in (V).

W. F. Eberlein (Madison, Wis.).

Nikodým, Otton Martin. Remarques sur la pseudo-topologie et sur les fonctionnelles linéaires. C. R. Acad. Sci. Paris **229**, 863–865 (1949).

Some remarks on pseudo-topologies [cf. the three preceding reviews]. The author notes that p.t. were first considered by A. D. Alexandroff [Rec. Math. [Mat. Sbornik] N.S. **8(50)**, 307–348 (1940); **9(51)**, 563–628 (1941); **13(55)**, 169–238 (1943); these Rev. **2**, 315; **3**, 207; **6**, 275] but were applied to representations of linear functionals as integrals only with respect to "charges," i.e., Jordan measures on $B(G)$ satisfying a regularity condition. The regularity ensures uniqueness of the Alexandroff integral representations, while, for example, a bounded linear functional on $C(0, 1)$ has at least $2^{\aleph_0}$ Fréchet integral representations. The concept of p.t. can be extended on replacing subsets of S by elements of a Boolean algebra with unit. A proof of such a somatic extension of the Bochner and Phillips [Ann. of Math. (2) **42**, 316–324 (1941); these Rev. **2**, 315] finitely-additive analogue of the Radon-Nikodým theorem is sketched.

W. F. Eberlein (Madison, Wis.).

Iseki, Kiyoshi. General analysis in abstract space. II. J. Osaka Inst. Sci. Tech. Part I. **1**, 89–90 (1949).

[For part I see Shimoda and the author, same vol., 61–66 (1949); these Rev. **11**, 368.] Suppose D is an open set in a complex Banach space E. Let $E_1, E_2, \cdots$ be a sequence of complex Banach spaces, and let E^* be the Banach space of sequences $y=(y_1, y_2, \cdots)$, $y_i\epsilon E_i$, with $\|y\|^p=\sum_{i=1}^{\infty}\|y_i\|^p<\infty$, where $p>1$. Let $f(x)=(f_1(x), f_2(x), \cdots)$ be a function defined on D to E^*. The assertion is that f is analytic on D if and only if it is continuous and each f_i is analytic on D. The necessity of the conditions is obvious. The proof of the sufficiency, as given, is inadequate, but the deficiencies are easily removed. There are some obvious misprints, and a minor defect in the definition of analyticity.

A. E. Taylor (Los Angeles, Calif.).

Ghika, Al. On reflexive Banach spaces. Acad. Repub. Pop. Române. Bul. Şti. A. **1**, 639–644 (1949). (Romanian. Russian and French summaries)

Let E be a Banach space, E^* its conjugate space. According to the Hahn-Banach theorem, there exists $k_x^*\epsilon E^*$ such that $k_x^*(x)=\|x\|$ and $\|k_x^*\|=1$. The author shows the following two statements for a reflexive space E: (1) For any $x^*\epsilon E^*$ one may write $x^*(x)=\lim_{n\to\infty}\sum_{i=1}^{r_n}\alpha_{ni}k^*_{x_{ni}}(x)$; (2) if $k^*_{\alpha x}=\alpha k_x^*/|\alpha|$ for $x\neq\theta$, $\alpha\neq 0$, if the set $\{\alpha k_x^* \mid \|x\|=1, -\infty<\alpha<\infty\}$ is linear, and if k_x^* is weakly (i.e., weak-weak) continuous in x for $\|x\|=1$, then, for any $x^*\epsilon E^*$, $x^*(x)=\|x^*\|k^*_{x_0}(x)$ for some x_0.

J. V. Wehausen.

Ghika, Al. On Banach spaces with a differentiable norm. Acad. Repub. Pop. Române. Bul. Şti. A. **1**, 645–653 (1949). (Romanian. Russian and French summaries)

Let E be a Banach space. The author proves the following. (1) If $\lim_{\zeta\to 0}[\|x+\zeta y\|-\|x\|]/\zeta=\delta N(x, y)$ exists for each pair $x, y\epsilon E$, excluding $x=\theta$, and if $\delta N(x, y)$ is continuous

in y for each $x \neq \theta$, then $\|x\| = \delta N(x, x)$ and $\delta N(x, y)$ is a linear continuous functional in y for each $x \neq \theta$, is of norm 1 as an element of E^*, and is homogeneous of degree 0 in x. (2) A theorem analogous to the second one stated in the preceding review, but with $\delta N(x, y)$ replacing $k_x^*(y)$.

J. V. Wehausen (Providence, R. I.).

Ghika, Al. On the problem of moments. Acad. Repub. Pop. Române. Bul. Şti. A. 1, 671–679 (1949). (Romanian. Russian and French summaries)

The author wishes to carry over into a reflexive Banach space with differentiable norm the method used by F. Riesz [Systèmes d'équations linéaires à une infinité d'inconnues, Gauthier-Villars, Paris, 1913; pp. 48–63] to solve the moment problem in l_p, $p>1$ [i.e., to prove theorem 5, p. 57, of Banach, Théorie des opérations linéaires, Warsaw, 1932]. The stated purpose is to avoid the use of the Hahn-Banach theorem and to solve the problem effectively. Uniqueness conditions are also given. *J. V. Wehausen.*

Akhyèser, N. On a class of integral operators. Akad. Nauk Ukrain. RSR. Zbirnik Prac' Inst. Mat. **1946**, no. 8, 111–120 (1947). (Ukrainian. Russian and English summaries)

A Carleman kernel [see M. H. Stone, Linear Transformations in Hilbert Space . . . , Amer. Math. Soc. Colloquium Publ., v. 15, New York, 1932, chapter 10] $K(s, t)$ $(-\infty < s, t < \infty)$ is said to be a B-kernel, if there exists a measurable function $M(s) \geqq 0$ such that

$$|K(s, t)| \leqq M(s) \cdot M(t)$$

almost everywhere. Let $P(s)$ be measurable and greater than 0 almost everywhere; denote by L_P the set of all complex measurable functions $f(s)$ such that $\int_{-\infty}^{\infty} P(s)|f(s)|\,ds < \infty$. Then L_P is a Banach space with norm $\|f\|_P = \int_{-\infty}^{\infty} P(s)|f(s)|\,ds$. The author proves the following theorem: If T is a Hermitian operator in L^2 with domain $\mathfrak{D}T$ everywhere dense in L^2, then the following are equivalent: (1) T is an integral operator with a B-kernel; (2) there exists a measurable function $P(s) > 0$ almost everywhere in $(-\infty, \infty)$ such that (i) $L^2 \cap L_P \subseteq \mathfrak{D}T^*$, where T^* is the adjoint of T, (ii) $|(T^*f, g)| \leqq \|f\|_P \cdot \|g\|_P$, (iii) $|(T^*f, T^*g)| \leqq \|f\|_P \|g\|_P$, (iv) $(T^*f, g) = (f, T^*g)$ for all $f, g \in L^2 \cap L_P$. This theorem enables the author to improve parts of the proof of the fundamental theorem on Carleman kernels [see Stone, loc. cit., theorem 10.4]. In the author's proof of the theorem the usual function $K(s)$ $(= [\int_{-\infty}^{\infty} |K(s, t)|^2 dt]^{\frac{1}{2}}$ whenever significant and 0 elsewhere) is used. To prove that (1)→(2) the operator $A^0 f = \int_{-\infty}^{\infty} K(s, t) f(t)\,dt$ for $f \in L^2 \cap L_K$ is considered, and $P(s)$ is defined as max $[1, K(s), M(s)]$. The proof of (2)→(1) is more difficult. Let $\mathfrak{M}$ be the set of all functions $\sum_{i=1}^{n} c_i h_P(s, \Delta_i)$, $n = 1, 2, \cdots$, where c_i are complex constants with rational real and imaginary parts, Δ_i an interval with rational extremities, $h_P(s, \Delta) = p(s)$ for $s \in \Delta$ and $h_P(s, \Delta) = 0$ for $s \bar{\epsilon} \Delta$, $p(s) = 1/P(s)$ for $P(s) \geqq 1$ and $p(s) = 1$ for $P(s) < 1$. The set $\mathfrak{M}$ is denumerable and everywhere dense in L^2 and in L_P. Now let $f, g \in L^2 \cap L_P$; (g, T^*f) is a continuous linear functional of g in $L^2 \cap L_P$. Using the Hahn-Banach extension theorem and integral formulas for linear functionals, it is proved that there exists $K(s, t)$ such that $T^*f = \int_{-\infty}^{\infty} K(s, t) f(t)\,dt$ for every $f \in \mathfrak{M}$, and every s which does not belong to a certain null set depending on f. This formula can be extended over the whole L_P. This gives the inequality $|K(s, t)| \leqq P(s) \cdot P(t)$ almost everywhere. The inequality $\int_{-\infty}^{\infty} |\int_t^{t+\delta} K(s, u) p(u)\,du|^2 ds \leqq \delta^2$ and an application of known theorems on the derivative of the integral prove that $K(s, t)$ is a Carleman B-kernel attached to T. The author cites, besides the known books on Carleman kernels, also Neumark [Bull. Acad. Sci. URSS. Sér. Math. [Izvestiya Akad. Nauk SSSR] **4**, 53–104, 277–318 (1940); these Rev. **2**, 104, 105]. *O. Nikodým* (Gambier, Ohio).

Povzner, A. On some applications of a class of Hilbert spaces of functions. Doklady Akad. Nauk SSSR (N.S.) **74**, 13–16 (1950). (Russian)

The author applies the considerations of his preceding note (I) [same Doklady (N.S.) **68**, 817–820 (1949); these Rev. **11**, 372] to a number of special spaces, and derives some of the known criteria for sets of uniqueness and for solutions of equations. If H is the space of functions analytic inside a circle Γ with $\int_\Gamma |f|^2 < \infty$, then the function $G(z, \zeta)$ of the note cited above is $(1 - z\bar{\zeta})^{-1}$ and the determinant conditions of (I) for $a_1, \cdots, a_n, \cdots$, to be a uniqueness set reduce to conditions of Blaschke and of Walsh [Interpolation and Approximation by Rational Functions . . . , Amer. Math. Soc. Colloquium Publ., v. 20, New York, 1935, p. 325]. In the space of functions analytic inside Γ with $\iint |f|^2 < \infty$, $G(z, \zeta) = (1 - z\bar{\zeta})^{-2}$. These spaces with simple, closed, piecewise smooth Γ are still g-spaces and the G-function of the second space is the kernel function of S. Bergmann [Mitt. Forsch.-Inst. Math. Mech. Univ. Tomsk 1, 242–256 (1937)].

If $\tau(\rho)$ is continuous for $0 \leqq \rho < \infty$, if $\tau(0) = 0$, and $\int_0^\infty e^{-\tau(\rho)} \rho^n\,d\rho < \infty$ for $n = 0, 1, \cdots$, define the scalar product $(f, g) = (2\pi)^{-1} \int_0^{2\pi} e^{-\tau(\rho)} f(\rho e^{i\theta}) g(\rho e^{i\theta})\,d\rho\,d\theta$, and let H_τ be the class of all functions analytic in the whole plane for which $(f, f) < \infty$. From the preceding paper and completeness of $1, z, z^2, \cdots$ in H_τ, $G(z, \zeta) = \sum z^k \bar{\zeta}^k / C_k$ $(= \psi(z\bar{\zeta}))$, where

$$C_k = \int_0^\infty e^{-\tau(\rho)} \rho^{2k}\,d\rho.$$

[When $\tau(\rho) = 2p\rho$ $(p > 0)$, $\psi(z) = 2p \cosh h(2pz^{\frac{1}{2}})$, and H_τ includes the class of functions of exponential type $< p$.] Then (I) gives theorem 1: If the sequence $a_1, a_2, \cdots$ is a uniqueness set for the space H_τ, then each $f(z)$ in H_τ can be calculated by countable processes from its values at the points a_n. An application to Newtonian interpolation yields some results of Nörlund [Leçons sur les séries d'interpolation, Gauthier-Villars, Paris, 1926] and application to quasi-analytic functions yields a criterion for quasi-analyticity due to Carleman [Ark. Mat. Astr. Fys. 17, no. 9 (1923)].

M. M. Day (Urbana, Ill.).

Maruyama, Gisiro. Notes on Wiener integrals. Kōdai Math. Sem. Rep. **1950**, 41–44 (1950).

The author extends the Cameron-Martin translation theorem for Wiener integrals to the case of the translation $T(x) = x - x_0$, where $x_0(t)$ is an absolutely continuous function vanishing at the origin and having a derivative of class L_2. [The necessary hypothesis of absolute continuity is here supplied by the reviewer.] He also simplifies the Cameron-Hatfield theorem which expresses the value of a bounded measurable functional at a point of mean continuity as a certain infinite dimensional Abel sum of one of its orthogonal developments in Fourier-Hermite functionals. Finally, the author shows that the orthogonal development is almost everywhere Abel summable to the functional in an appropriate sense, even without a continuity condition.

R. H. Cameron (Minneapolis, Minn.).

Nemyckiĭ, V. V. The structure of one-dimensional limiting integral manifolds in the plane and three-dimensional space. Vestnik Moskov. Univ. **1948**, no. 10, 49–61 (1948). (Russian)

This paper concerns continuous stationary flows in locally compact metric spaces E, i.e., general dynamical systems [cf. G. D. Birkhoff, Dynamical Systems, Amer. Math. Soc. Colloquium Publ., v. 9, New York, 1927]. A limiting integral manifold of a flow is defined as an invariant set K having a neighborhood (region of attraction) every point of which has K as set of ω-limiting points or as set of α-limiting points. If, further, there is a $\delta(\epsilon)>0$ defined for $\epsilon>0$ such that points within the δ-neighborhood of K never leave the ϵ-neighborhood of K with increasing time, then K is called asymptotic; the author proves that, if K is compact, this is equivalent to requiring the existence of a $T(d)$, defined for $d>0$, such that every point of a chosen region of attraction is transformed by the flow into points of the d-neighborhood of K for all times greater than T. The author considers cross-sections of such flows [cf. H. Whitney, Annals of Math. (2) **34**, 244–270 (1933); Duke Math. J. **4**, 222–226 (1938)], points out several simple properties, and then proves that for every flow in the plane there is a cross-section which is an arc through every regular point [the reviewer considers the long proof given for this unnecessary in view of the results of Whitney in the papers cited above]. It is then shown that every limiting integral manifold of a flow in the plane is asymptotic and is a simple closed curve. It is indicated by an example that a limiting integral manifold for a flow in 3-space need not be asymptotic and it is proved that every compact one-dimensional asymptotic limiting integral manifold K in space is a simple closed curve, provided that K is a minimal set and that the flow on K is uniformly sectioned with respect to some point p_0. This last condition requires that there be a cross-section S through p whose transforms under the flow do not meet S for suitably chosen but arbitrarily large values of the time t.

W. Kaplan (Ann Arbor, Mich.).

Utz, W. R. Unstable homeomorphisms. Proc. Amer. Math. Soc. **1**, 769–774 (1950).

Let X be a metric space with metric d and let f be a homeomorphism of X onto X. The following definitions are made. The homeomorphism f is unstable provided there exists $a>0$ such that $x, y\epsilon X$ with $x\neq y$ implies $d(f^n x, f^n y)>a$ for some integer n (for example, the shift transformation of symbolic dynamics is unstable). If $x, y\epsilon X$, then x and y are positively {negatively} asymptotic provided that $x\neq y$ and $d(f^n x, f^n y)\to 0$ as $n\to+\infty$ $\{n\to-\infty\}$; x and y are unilaterally asymptotic in case x and y are positively or negatively asymptotic. Let X be compact and let f be unstable. The following theorems are proved. Every nonzero power of f is unstable (whence the periodic points of X with a given period form a finite set, and the periodic points of X form a countable set). If X is self-dense, then there exists a unilaterally asymptotic pair of points of X. If X is self-dense and if $x\epsilon X$ is periodic, then there exists $y\epsilon X$ such that x and y are unilaterally asymptotic.

W. H. Gottschalk (Philadelphia, Pa.).

Mathematical Statistics

Fix, Evelyn. Tables of noncentral χ^2. Univ. California Publ. Statist. **1**, 15–19 (1949).

To find the power function of the test that $\theta_i=\theta_i^0$, $1\leqq i\leqq f$, where X_i are independently normally distributed with mean θ_i and variance 1, tables of noncentral χ^2 are needed. Two tables are constructed corresponding to critical sizes $\alpha=.01$ and .05. These tables give $\lambda=\sum_{i=1}^{f}(\theta_i-\theta_i^0)^2$ as a function of f and the probability of error β. The value of λ is given to three or four decimal places and $f=1(1)20(2)40(5)60(10)100$ and $\beta=.1(.1).9$, where $x=a(b)c$ indicates that x goes from a to c by steps of size b. *H. Chernoff* (Urbana, Ill.).

Hartley, H. O., and Pearson, E. S. Tables of the χ^2-integral and of the cumulative Poisson distribution. Biometrika **37**, 313–325 (1950).

A working table for $P(\chi^2, \nu)=2^{-\frac{1}{2}\nu}\Gamma(\frac{1}{2}\nu)^{-1}\int_{\chi^2}^{\infty}e^{-\frac{1}{2}x}x^{\frac{1}{2}\nu-1}dx$ to five decimal places is given for $\nu=1(1)20(2)70$ and $\chi^2=0.000(0.001)0.01(0.01)0.1(0.1)2.0(0.2)10.0(0.5)20(1)40(2)134$. This table gives values of the cumulative Poisson distribution since $P(\chi^2, \nu)=\sum_{i=0}^{c-1}e^{-m}m^i/i!$ with $m=\frac{1}{2}\chi^2$ and $c=\frac{1}{2}\nu$. Methods of interpolation are suggested.

H. Chernoff (Urbana, Ill.).

Bosse, Lothar, Linder, A., Ludwig, W., and Vajda, Stefan. Deutsche Bezeichnungen für Fachausdrücke der mathematischen Statistik. Mitteilungsblatt Math. Statist. **2**, 138–144 (1950).

Extracts from letters from the several authors of the title concerning possible German equivalents for English technical terms in statistics.

Stuart, A. The cumulants of the first n natural numbers. Biometrika **37**, 446 (1950).

The author expands the logarithm of the known moment generating function of the discontinuous rectangular distribution formed by the first n natural numbers, obtaining an explicit general formula for the cumulants.

C. C. Craig (Ann Arbor, Mich.).

Silverstone, H. A note on the cumulants of Kendall's S-distribution. Biometrika **37**, 231–235 (1950).

The author derives the general expression for cumulants of Kendall's $S=\frac{1}{2}n(n-1)-2I$, where I is the number of inversions in order of the first n natural numbers, all permutations being assumed equally likely. Information as to approach to normality of the distribution of S with increasing n is then obtained by an examination of the behavior of the coefficients in the expansion of the cumulant generating function and bounds to the error resulting from the use of the normal frequency and distribution functions as approximations to the corresponding functions for S are shown to be $O(1/n)$. Further approximations by means of the first three terms of the corresponding type A series are found and comparisons between the two modes of approximation and the exact values are made for several values of n.

C. C. Craig (Ann Arbor, Mich.).

Freeman, Murray F., and Tukey, John W. Transformations related to the angular and the square root. Ann. Math. Statistics **21**, 607–611 (1950).

Approximations to K defined by

$$(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{K}e^{-\frac{1}{2}t^2}dt=\text{Prob}\,[x\leqq k\,|\,n, p],$$

where x is distributed as a binomial variable with parameters n and p, are considered. Transformations which stabilize the variance of binomial and Poisson variables are given. The simplest useful approximations provided by the authors are: $K\approx 2[(k+1)q]^{\frac{1}{2}}-2[(n-k)p]^{\frac{1}{2}}$ and

$$y=[n+\tfrac{1}{2}]^{\frac{1}{2}}\{\arcsin\,[x/(n+1)]^{\frac{1}{2}}+\arcsin\,[(x+1)/(n+1)]^{\frac{1}{2}}\}$$

for stable variance (≈ 1). In the case of a Poisson variable this last becomes $[(x)^{\frac{1}{2}}+(x+1)^{\frac{1}{2}}]$. Other more accurate approximations are given. *M. A. Woodbury.*

✱Springer, Melvin Dale. Joint Sampling Distribution of Mean and Standard Deviation for a Chi-Square Universe. Abstract of a Thesis, University of Illinois, 1950. 16 pp.

The author derives the joint sampling distribution of the mean and standard deviation in samples of N from a χ^2 distribution with two degrees of freedom. The regression functions of the joint distribution are investigated as well as the correlation between the mean and standard deviation. Similar properties are derived for the χ^2 distribution with four degrees of freedom for sample sizes of 3 and 4.

L. A. Aroian (Culver City, Calif.).

David, F. N. An alternative form of χ^2. Biometrika **37**, 448–451 (1950).

Let $\chi_1^2=\sum_{i=1}^{k}(n_i-Np_i)^2/n_i$, where the n_i have a multinomial distribution with parameters $N, p_1, \cdots, p_k$. By formally expanding χ_1^2 and χ_1^4 and taking expectations, expressions are obtained which are written as expansions in powers of N^{-1} of the moments of first and second order of χ_1^2. (Actually these moments are infinite, though the expansions can be interpreted as approximating the moments of a random variable which has the same asymptotic distribution as χ_1^2.) The author draws the conclusion that "the quantity χ_1^2 should not be used instead of χ^2 for testing goodness of fit." *W. Hoeffding* (Chapel Hill, N. C.).

Fraser, D. A. S. Note on the χ^2 smooth test. Biometrika **37**, 447–448 (1950).

Seal [Biometrika **35**, 202 (1948); these Rev. **11**, 42] stated, in a somewhat ambiguous form, a theorem on the independence of the sum of squares of normal variables with linear constraints and the signs of the variables. The author gives a "corrected statement" of Seal's theorem and discusses applications to testing goodness of fit. [For another revised version of Seal's theorem see the review of Seal's paper.] *W. Hoeffding* (Chapel Hill, N. C.).

Rasch, G. A vectorial t-test in the theory of normal multivariate distributions. Mat. Tidsskr. B. **1950**, 76–81 (1950).

Starting from the Wishart distribution, using only elementary transformations but making full use of vector and matrix operations, a neat derivation of the distribution of Hotelling's "generalized" T is given. Some related distributions are also found in the course of the derivation.

D. G. Chapman (Seattle, Wash.).

Westenberg, J. The median and interquartile range test applied to frequency distributions plotted on a circular axis. Nederl. Akad. Wetensch., Proc. **53**, 1034–1037 = Indagationes Math. **12**, 378–381 (1950).

Hyrenius, Hannes. Distribution of 'Student'-Fisher's t in samples from compound normal functions. Biometrika **37**, 429–442 (1950).

Let $\phi(x;\mu,\lambda)$ denote the distribution of a normal variable with mean μ and variance λ. The author investigates the distribution of Student's t-statistic in random samples from populations having the compound normal distribution $\sum p_i\phi(x;\mu_i,\lambda_i)$ with $\sum p_i=1$, $0<p_i<1$. An important conclusion is that it is not sufficient to measure departure from normality of the underlying distribution by the first two β-coefficients or their equivalents as has usually been done in experimental sampling investigations of the problem. The structure of the underlying variation also has to be considered. The results of three sampling experiments are compared with the theoretical findings, showing on the whole good agreement. *G. E. Noether* (New York, N. Y.).

Gayen, A. K. The distribution of the variance ratio in random samples of any size drawn from non-normal universes. Biometrika **37**, 236–255 (1950).

The paper investigates the distribution of the variance ratio used for testing (i) the homogeneity of means in one-way classifications and (ii) the compatibility of two variances in samples from populations, the distributions of which are given by the first four terms of the Edgeworth series

$$f(x)=\phi(x)-\frac{\lambda_3}{3!}\phi^{(3)}(x)+\frac{\lambda_4}{4!}\phi^{(4)}(x)+\frac{\lambda_3^2}{72}\phi^{(6)}(x).$$

Formulas for the upper tail probabilities in terms of λ_3^2 and λ_4 are given and corrective functions for determining these probabilities at the upper 5% normal-theory significance levels have been tabulated for degrees of freedom $\nu_1=1(1)6$, 8, 12, 24, ∞ and $\nu_2=1(1)6$, 8, 12, 20, 24, 30, 40, 60, 120, ∞. The author indicates that the formulas are valid asymptotically for any form of universe and expresses the belief that for moderate sample sizes they have "quite an extended range of applicability." This latter statement should be taken with some caution, particularly in view of a similar investigation by Hyrenius [see the preceding review] into the distribution of Student's t-statistic in samples from non-normal universes, since this author comes to the conclusion that the quantities λ_3 and λ_4 are not sufficient to characterize the underlying distribution for purposes of his investigation.

G. E. Noether (New York, N. Y.).

Gayen, A. K. Significance of difference between the means of two non-normal samples. Biometrika **37**, 399–408 (1950).

The distribution of the ratio of the difference in the means to the estimate of the standard error of the difference based on the pooled sum of squares is studied for samples of sizes n_1 and n_2 from two populations having the same variance when the populations are adequately represented by the first four terms of Edgeworth's form of the Charlier Type A series. One of the author's conclusions is that the formulas given will be widely applicable even though normal theory does not yield an accurate answer in certain cases. Approximate corrections to the one-tail $2\frac{1}{2}$% significance level of Student's statistic are tabulated for 2(2)8, 12, 20, 24, 30, 40, 60, and 120 degrees of freedom for the case when the samples are of equal size. *M. A. Woodbury* (Princeton, N. J.).

Hartley, H. O. The maximum F-ratio as a short-cut test for heterogeneity of variance. Biometrika **37**, 308–312 (1950).

The author proposes the maximum ratio of sample variances as a test statistic for the hypothesis that the population variances are equal, and provides approximate 5% points for the test in the case when all sample variances have the same number of degrees of freedom. He investigates power by an approximate method, under the assumption that the population variances are themselves sampled from a log-normal distribution, and finds his test to be nearly as powerful as the more complicated Bartlett M test, when the number of variances is not over 12. *J. L. Hodges, Jr.*

Ehrenberg, A. S. C. The unbiased estimation of heterogeneous error variances. Biometrika 37, 347–357 (1950).

The model considered is $x_{ij}=\alpha_i+\beta_j+\epsilon_{ij}$, $i=1, \cdots, p$; $j=1, \cdots, q$, where x_{ij} is the observation of the jth observer on the ith item, α_i is the constant common to all observations on the ith item, β_j is the bias of the jth observer, and ϵ_{ij} is the (residual) error, assumed to be distributed with mean 0 and variance σ_j^2. It is shown that maximum likelihood estimates of the variances, derived under the assumption of normality of errors, are biased in general and are not consistent (as $p\rightarrow\infty$). Quadratic forms in x_{ij} that are unbiased estimates of the variances are studied. Efficiency of estimates, validity of certain t- and F-tests, and related problems are considered. *T. W. Anderson.*

Hughes, Harry M. Estimation of the variance of the bivariate normal distribution. Univ. California Publ. Statist. 1, 37–51 (1949).

The author's summary is as follows: "Let X_1 and X_2 be two random variables normally distributed with known means m_1 and m_2 and with unknown variance σ^2. Consider an experiment in which the observable variable is $Y=[(X_1-m_1)^2+(X_2-m_2)^2]^{\frac{1}{2}}$, the distribution of which is given by

$$P\{Y<y\}=\begin{cases}1-e^{-y^2/2\sigma^2}, & y\geqq 0\\ 0, & y<0.\end{cases}$$

The paper considers the problem of estimating σ when the observations are grouped, obtains a solution by the method of minimum reduced chi-square with linear restrictions, and derives the asymptotic variance of the estimate. It then finds the optimum grouping for known σ, and considers the optimum grouping, for two and three groups, when only a certain range of σ is known. If such range is small, it shows that the upper bound should be used for obtaining the optimum grouping; and finally, the effect of using non-optimum groupings is exhibited by table and chart." *G. E. Noether* (New York, N. Y.).

Nagler, H. On the best unbiased quadratic estimate of the variance. Biometrika 37, 444–445 (1950).

The author derives a best estimate (in the sense of variance) for the variance of a distribution, as a function of a not necessarily independent sample; in the independent case his estimate reduces to the classical one. Although at one point the author claims that his results do not "exact any limitations on the nature of the distributions," it seems to the reviewer that the secret of the success of the author's combinatorial methods is in the tacit assumption that the distribution that is estimated is discrete. *P. R. Halmos.*

Cohen, A. C., Jr. Estimating the mean and variance of normal populations from singly truncated and doubly truncated samples. Ann. Math. Statistics 21, 557–569 (1950).

This paper deals with the estimation of the mean and variance of a normal population from singly or doubly truncated samples. For doubly truncated samples three cases are treated, viz., the number of unmeasured observaions is unknown, the number of unmeasured observations in each tail is known, and only the total number of unmeasured observations in both tails is known. Singly truncated samples are then handled as special cases of these. The maximum likelihood estimation equations are found and iterative methods using standard tables for solving them are given and illustrated. Finally, asymptotic variances and covariances of these estimates are found from the information matrices. *C. C. Craig* (Ann Arbor, Mich.).

Kawada, Yukiyosi. Independence of quadratic forms in normally correlated variables. Ann. Math. Statistics 21, 614–615 (1950).

Given two quadratic forms in the same set of normally correlated variables, the author gives a short proof of four simple conditions for the product of the two variance matrices to be zero, which are then conditions for the forms to be independent. The necessity of the condition that the product of the variance matrices be zero for independence of the forms follows readily. If one of the forms is non-negative, he shows the first four conditions can be reduced to two. *C. C. Craig* (Ann Arbor, Mich.).

Baker, A. G. Properties of some tests in sequential analysis. Biometrika 37, 334–346 (1950).

This paper is concerned with the problem of deciding between two univariate normal distributions with a common mean σ^2, by means of the sequential probability ratio test invented by Wald in 1943 [cf., e.g., Ann. Math. Statistics 16, 117–186 (1945); these Rev. 7, 131]. The first part of the paper describes a sampling experiment designed to check certain approximations given by Wald for the case when σ^2 is known. The second part of the paper is concerned with the case when σ^2 is unknown. The author investigates a procedure where the Wald test is followed formally, except that the unknown value of σ^2 is replaced by s^2, an independent estimate of σ^2 based on f degrees of freedom. More precisely: Let μ_0 and $\mu_1>\mu_0$ be the two normal means, and α and β the desired probabilities of errors of the first and second kind. For σ^2 known the Wald test proceeds by using $\sigma^2\log A/(\mu_1-\mu_0)$ and $\sigma^2\log B/(\mu_1-\mu_0)$ as bounds for the cumulative sums. Here A and B are simple expressions in α and β. The author proposes to use as bounds in the case where σ^2 is unknown the quantities $s^2C/(\mu_1-\mu_0)$ and $-s^2D/(\mu_1-\mu_0)$, where C and D are positive constants. Using Wald's results and taking expected values with respect to s^2 the author obtains the operating characteristic curve and average sample number of this test. For some given f, $\alpha=\beta$, the author constructs a table of $C=D$ corresponding to these values; for $\alpha\neq\beta$ he suggests a rough approximation. *J. Wolfowitz* (New York, N. Y.).

Kemperman, J. H. B. Some methods from sequential analysis. II. Math. Centrum, Amsterdam. Rapport ZW-1950-003, 29 pp. (1950). (Dutch)

For part I see Rapport **ZW-1949-009** in the same series; these Rev. 11, 449.

★Kemperman, Johannes Henricus Bernardus. The General One-Dimensional Random Walk with Absorbing Barriers with Applications to Sequential Analysis. Excelsiors Foto-Offset, 's-Gravenhage, 1950. vii+111+4 pp.

Motivated by Wald's sequential analysis, the author studies chiefly the one-dimensional random walk with two constant absorbing barriers, including the case where one barrier is at infinity. Chapter 1 gives, formally, the probabilities of absorption at the nth step. Chapter 2 gives new results concerning Wald's fundamental identity. Chapter 3 applies the fundamental identity to obtain approximations to the probabilities of error and the expected number of observations in Wald's sequential probability ratio test. Chapters 4 and 5 discuss other methods of obtaining these

quantities. Chapter 6 discusses these quantities for the case when one barrier is at $+\infty$ and the identically distributed chance variables are bounded below. Finally, chapter 7 discusses the following random walk: The chance variables are identically distributed and can take only a finite number of integral values. At various lattice points there are probabilities of absorption; the barriers are now of a chance character. The author obtains various probabilities of absorption. The results of the paper do not easily lend themselves to summary.

J. Wolfowitz (New York, N. Y.).

Neyman, J. On the problem of estimating the number of schools of fish. Univ. California Publ. Statist. **1**, 21–36 (1949).

Suppose that a ship scouts for schools of fish so that at any time t it has a certain chance to discover a school; after every discovery the ship spends a fixed time h on catching the school and during this time no scouting occurs. The problem is essentially that of counters, except that more complicated assumptions on the incoming traffic are appropriate since this depends on the geographic distribution of schools and since various ships are not statistically independent. The author discusses effective methods of estimating the number of schools from meager records, and the problem of which data are most efficient.

W. Feller (Princeton, N. J.).

Chandler, K. N. On a theorem concerning the secondary subscripts of deviation in multivariate correlation using Yule's notation. Biometrika **37**, 451–452 (1950).

Consideration of a specific example shows that the following theorem [e.g., Yule and Kendall, An Introduction to the Theory of Statistics, 11th ed., Griffin, London, 1937, p. 265] is incorrect: The product-sum of any two deviations is unaltered by omitting any or all of the secondary subscripts of either which are common to the two.

T. W. Anderson (New York, N. Y.).

Kishen, K. Expression of unitary components of the highest order interactions in 3^5, 3^6, 4^4 and 5^3 designs in terms of sets for these interactions. J. Indian Soc. Agric. Statistics **2**, 196–211 (3 plates) (1950).

In a previous paper [Sankhyā **6**, 133–140 (1942); these Rev. **4**, 281] the author has given a method to express every single degree of freedom in a factorial S^m (S a prime power) design in terms of its sets for main effects and interactions. In the present paper the author shows that these expressions become particularly simple if the degree of freedom to be expressed belongs itself to an interaction. The expressions are explicitly obtained and tabulated for some of the interactions of a 3^5, 3^6, 4^4, and 5^3 design. *H. B. Mann.*

Taylor, J. The comparison of pairs of treatments in split-plot experiments. Biometrika **37**, 443–444 (1950).

With certain pairs of treatment means in a split-plot experiment, the error variance of their difference is a linear function of the two basic error variances in the design. The ratio of the difference to its estimated standard error does not follow a t-distribution. An approximate t-test is proposed, by assigning an effective number of degrees of freedom to the standard error of the difference. The rule for assignment is that proposed by Welch [Biometrika **36**, 293–296 (1949); these Rev. **11**, 527] and Satterthwaite [Biometrics Bull. **2**, 110–114 (1946)]. *W. G. Cochran.*

Howell, John M. Errata to "Control chart for largest and smallest values." Ann. Math. Statistics **21**, 615–616 (1950).

See the same Ann. **20**, 305–309 (1949); these Rev. **10**, 724.

Ghosh, Birendranath. A multi-stage stochastic model for some natural fields. Calcutta Statist. Assoc. Bull. **3**, no. 9, 21–31 (1950).

Zadeh, Lotfi A., and Ragazzini, John R. An extension of Wiener's theory of prediction. J. Appl. Phys. **21**, 645–655 (1950).

A heuristic development is given for the following prediction problem. The message consists of two parts, a function of time representable as a polynomial of degree n about which only the degree n is assumed known, and a stationary random function of time. The weighting function of the predictor is required to vanish after a finite lag. Results of Wiener and of Phillips and Weiss are obtained as limiting cases.

N. Levinson (Cambridge, Mass.).

Theil, H. On "upcrosses" and "downcrosses" in time series. Math. Centrum Amsterdam. Rapport **ZW-1950-010**, 8 pp. (1950). (Dutch)

Reiersøl, Olav. Identifiability of a linear relation between variables which are subject to error. Econometrica **18**, 375–389 (1950).

The author considers observed variables x_1 and x_2 of the form y_i+v_i $(i=1, 2)$, where the y_i are true values, v_i are errors of observation, and the relation $y_2=\beta y_1+\beta_0$ holds. The linear structure so defined is said to be identifiable when the parameters in the joint distribution of the y_i and v_i in the conditions given are completely determined from a knowledge of the joint distribution of the x_i. After a brief review of earlier related work the author considers two models in both of which the variables v_i are independent of the variables y_i and in the one the v_i are jointly normally distributed while in the other the v_i are stochastically independent. In both cases he finds necessary and sufficient conditions that β be identifiable and then that the remaining parameters be identifiable if β is. *C. C. Craig.*

Arbey, Louis. Les erreurs d'observations considérées comme liées. Bull. Astr. (2) **14**, 75–144 (1949).

In many physical problems successive observations are not mutually independent, but are dependent to a more or less degree. In this paper the author investigates the question of determining the component of error due to chance if successive observations are drawn from (a) a simple Markoff process, (b) from a Markoff process of higher order. Among other things he finds out how the simple correlation coefficient between two observations in a sequence falls off as the distance between the observations increases. This result is already well known for a simple Markoff process. A numerical illustration is given at the end of the paper.

B. Epstein (Detroit, Mich.).

TOPOLOGY

✱Freudenthal, H. Examples of topological research. Zeven voordrachten over topologie. [Seven Lectures on Topology]. Centrumreeks, no. 1. Math. Centrum Amsterdam, pp. 1–25. J. Noorduijn en Zoon, Gorinchem, 1950. 6 florins. (Dutch)

Expository paper.

✱van Dantzig, D. Topologico-algebraic reconnoitering. Zeven voordrachten over topologie. [Seven Lectures on Topology]. Centrumreeks, no. 1. Math. Centrum Amsterdam, pp. 56–79. J. Noorduijn en Zoon, Gorinchem, 1950. 6 florins. (Dutch)

This is an exposition of some of the author's results in topological algebra [see Fund. Math. 15, 102–125 (1930); Math. Ann. 107, 587–626 (1932); Compositio Math. 2, 201–223 (1935); 3, 408–426 (1936); Ann. Sci. École Norm. Sup. (3) 53, 275–307 (1936)], kept on a slightly popular level, expressed in geographical language.

O. Todd-Taussky (Washington, D. C.).

✱de Groot, J. The dimension concept and dimension zero. Zeven voordrachten over topologie. [Seven Lectures on Topology]. Centrumreeks, no. 1. Math. Centrum Amsterdam, pp. 26–35. J. Noorduijn en Zoon, Gorinchem, 1950. 6 florins. (Dutch)

Expository paper.

✱van Heemert, A. Pathological curves. Zeven voordrachten over topologie. [Seven Lectures on Topology]. Centrumreeks, no. 1. Math. Centrum Amsterdam, pp. 36–55. J. Noorduijn en Zoon, Gorinchem, 1950. 6 florins. (Dutch)

Expository paper.

Iseki, Kiyoshi. Some remarks on well known theorems in topology. J. Osaka Inst. Sci. Tech. Part I. 1, 81–82 (1949).

Denjoy, Arnaud. Les points de ramification des continus. C. R. Acad. Sci. Paris 231, 1184–1186 (1950).

Proof that the set of all cut points of a continuum which separate the continuum into more than two components is countable. The author apparently is unaware of the fact that this result in more general form was published by Kuratowski and Zarankiewicz [Bull. Amer. Math. Soc. 33, 571–575 (1927)], by the reviewer [Trans. Amer. Math. Soc. 30, 597–609 (1928)], and also is a direct consequence of deeper results on the Menger-Urysohn order of the cut points of a connected set [see the reviewer, Analytic Topology, Amer. Math. Soc. Colloquium Publ., v. 28, New York, 1942; these Rev. 4, 86]. *G. T. Whyburn.*

Rodnyanskiĭ, A. M., and Kaščenko, Yu. D. On irreducible continua. Mat. Sbornik N.S. 26(68), 321–340 (1950). (Russian)

In this paper, a number of questions in the theory of irreducible continua are thoroughly explored. A topological space R is said by the authors to be locally connected at $x \epsilon R$ if for every neighborhood $U(x)$, there exists a neighborhood $V(x)$ such that for all $y \epsilon V(x)$, there exists a connected subset T of $U(x)$ containing both x and y. [Reviewer's note. This definition is at variance with commonly used terminology. See, e.g., Hausdorff, Mengenlehre, 3d ed., de Gruyter, Berlin-Leipzig, 1935, pp. 155–156. For the purposes of the present review, we shall designate the property described above as weak local connectedness at x.] The authors first investigate the relation between local connectedness and weak local connectedness; stating without proof that if $x \epsilon R$ admits a neighborhood at all the points of which there is weak local connectedness, then R is locally connected at x; showing by an example in the plane that a compact connected space may have a point of weak local connectedness which is not a point of local connectedness; and proving that if C is a compact Hausdorff space, irreducible between points a and b, then every point of weak local connectedness is also a point of local connectedness. Next, it is shown that a separable metric space, which is irreducibly connected between two points and is either weakly locally connected, or has at every point arbitrarily small neighborhoods with compact boundaries, is necessarily a topological image of the closed interval $[0, 1]$. The main results of the paper center on the following objects. Let C be a compact connected metric space irreducible between two points a and b. Let L be the set of points in C at which weak local connectedness obtains. Let $M = C \cap L'$; let $\mathfrak{L}$ be the system of components of L, and $\mathfrak{M}$ the system of components of M. A large, one is tempted to say exhaustive, array of facts concerning L, M, $\mathfrak{L}$, and $\mathfrak{M}$ is assembled. We mention the following. Every set in $\mathfrak{L}$ is either a point or the topological image of an interval (open, closed, or half-open). If $A \epsilon \mathfrak{L}$, then $A \cap A'^{-}$ contains at most two points. If $B \epsilon \mathfrak{M}$, then B contains at least two points, is locally compact, and has the property that every pair of points of B is contained in a compact connected subset of B. There are only a countable number of sets in $\mathfrak{L}$ which contain more than one point; there may be n ($n < \aleph_0$), $\aleph_0$, or $2^{\aleph_0}$ sets in $\mathfrak{L}$ containing exactly one point; and these are the only possible cardinal numbers. The cardinal number of $\mathfrak{M}$ is either finite, or $\aleph_0$, or $2^{\aleph_0}$. *E. Hewitt* (Seattle, Wash.).

Bing, R. H. Complementary domains of continuous curves. Fund. Math. 36, 303–318 (1949).

A finite collection G of mutually exclusive connected open subsets of M is a partitioning of M if the sum of the elements of G is dense in M. If each element of G is of diameter less than ϵ, G is an ϵ-partitioning. If each element of G has property S, it is an S-partitioning. If p and q belong to a connected set M, $E(M; p, q)$ denotes the greatest lower bound of the diameters of all connected subsets of M containing $p+q$.

An S-partitioning G of M is a brick partitioning if: (a) Each domain containing a point of M which is a limit point of each of two elements of G also contains a point of M which is a limit point of each of these same two elements of G but of no other element of G; (b) each element of G is uniformly locally connected under $E(M; x, y)$; (c) each boundary point in M of an element of G is a boundary point of another element of G. The elements of a brick partitioning G of space are packed in something like brick; that is, for each pair of elements of G whose boundaries intersect each other, there is a point on their common boundary that is not a boundary point of any other element of G. An advantage of such a partitioning is that two adjacent elements of G may be consolidated into one element so that the resulting partitioning is also a brick partitioning. It is shown that if M is a set with property S,

then there is a sequence $G_1, G_2, \cdots$ such that G_i is a brick $(1/i)$-partitioning of M and G_{i+1} is a refinement of G_i.

An extensive machinery of partitioning theorems is now developed and applied by the author in establishing the following results concerning the complements of continuous curves. Let M denote a compact locally connected continuum. (I) If p, q, and r are three points of an M which is not separated by any pair of its points, there is an arc from p to q in M such that the complement of this arc is connected, has property S and contains r. (II) If p and q are two points of M, there is an arc from p to q such that each complementary domain of this arc has property S and each such pair of complementary domains is separated in M by a pair of points. Furthermore, for each positive number ϵ there are only a finite number of these complementary domains of diameter more than ϵ. (III) Suppose W is a totally disconnected closed subset of M. Then there is a dendron T in M containing W such that each component of $M-T$ has property S and for each positive number ϵ there are no more than a finite number of such components of diameter more than ϵ. (IV) If M is nondegenerate and is not separated by any pair of its points, then either M is a simple closed surface or there is a simple closed curve J in M such that $M-J$ is connected and has property S.

The author proposes the following problem. For each closed set W in M whose complement has property S, does there exist a countable collection A of arcs such that $W+A^*$ is a continuous curve each of whose complementary domains has property S? (A^* denotes the sum of the elements of A.)

W. W. S. Claytor (Washington, D. C.).

Wallace, A. D. A theorem on endpoints. Anais Acad. Brasil. Ci. **22**, 29–33 (1950).

Let X be a compact Hausdorff space, X_0 a closed subset, and p a positive integer. A subset A of X is p-connected (mod X_0) if for each element e of the pth cohomology group, $H^p(A, A\cap X_0)$, of A (mod $A\cap X_0$) we have $q^*e=0$ in $H^p(B, B\cap X_0)$ for each compact subset B of A, where q^* is the homomorphism induced by the identity mapping q of B into A. A compact set A in X is a T^p (mod X_0) if for each closed subset B, $H^p(B, B\cap X_0)$ vanishes. A point y of X is a p-endpoint (mod X_0) if each open set containing y contains an open set S, containing y, with $\bar{S}-S$ a T^p (mod X_0). A set A in X is totally p-disconnected (mod X_0) if each compact set in A which is p-connected (mod X_0) is a T^p (mod X_0). The cohomology theory in these definitions is that of Spanier [Ann. of Math. (2) **49**, 407–427 (1948); these Rev. **9**, 523]. With these definitions, the following theorem is proved: If X is p-connected (mod X_0) and if E is a set of p-endpoints (mod X_0), then $X-E$ is p-connected (mod X_0) and E is totally p-disconnected (mod X_0).

E. G. Begle (New Haven, Conn.).

Stone, A. H. Incidence relations in multicoherent spaces. II. Canadian J. Math. **2**, 461–480 (1950).

[Part I appeared in Trans. Amer. Math. Soc. **66**, 389–406 (1949); these Rev. **11**, 45.] A study is made of the degree of multicoherence of a connected and locally connected normal T_1 space S using primarily a modified nerve concept defined in terms of decomposition systems of the set intersection of subsystems of a given finite covering system of S. Among other things it is shown that if the sets in the finite covering system are nonempty, closed, and connected and have finite incidence in the sense that the set intersection of each subsystem has only a finite number of components, then $r(N)\leqq r(M)\leqq r(S)$, where N and M denote the ordinary nerve and modified nerve respectively of the covering system and $r(x)$ denotes the degree of multicoherence in the sense of Eilenberg. Also, it is shown that even for spaces S of the generality considered in this paper, $r(S)$ is equal to the analytic degree of multicoherence $\rho(S)$ defined in terms of mappings onto the circle, a result previously known only for Peano spaces. Conditions are also formulated under which $r(M)$ and $r(S)$ are equal, involving the assumption of unicoherence on the sets of the covering system.

G. T. Whyburn (Charlottesville, Va.).

Simonsen, W. On single-valued mappings between two spaces which are single-valued images of the same third space. Mat. Tidsskr. B. **1950**, 38–41 (1950). (Danish)

Let X, Y, Z be topological spaces, of which X shall be compact and Y Hausdorff, and let f be a Y-valued function defined on X and h a Z-valued function defined on Y. A sufficient condition for continuity of h is continuity of f and continuity of hf. *R. H. Fox* (Princeton, N. J.).

Yajima, Takeshi. Concerning the extremal point locus of 2-dimensional closed manifolds. J. Osaka Inst. Sci. Tech. Part I. **1**, 135–145 (1949).

The author considers a metrically convex, 2-dimensional, closed manifold R in which points sufficiently close together are joined by unique metric segments and prolongations of segments are unique whenever they exist. If $a\epsilon R$ and S_a is a metric segment issuing from a, then R contains a point b beyond which S_a cannot be prolonged (this is equivalent to the nonexistence of a point c of R such that b is metrically between a and c). The author shows that the locus E_a of such extreme points b (with respect to a) is connected, locally connected, arcwise connected, and has dimension at most 1. *L. M. Blumenthal* (Columbia, Mo.).

Trevisan, Giorgio. Una osservazione sul problema dei quattro colori. Rend. Sem. Mat. Univ. Padova **19**, 103–107 (1950).

It is shown that the problem of coloring a cubic map of n regions in four colors is equivalent to finding two solutions (x') and (x'') of a certain system of $2n-5$ congruences of the form $x_i+x_j+x_k\equiv 1$ (mod 2) in $3n-6$ unknowns, $x_1, \cdots, x_{3n-6}$, such that $x_i'x_i''\equiv 0$ (mod 2), $i=1, \cdots, 3n-6$. The author, however, formulates his ideas in terms of the dual map.

D. C. Lewis (Baltimore, Md.).

Fenchel, W. Remarks on finite groups of mapping classes. Mat. Tidsskr. B. **1950**, 90–95 (1950). (Danish)

Nielsen proved [Acta Math. **75**, 23–115 (1943); these Rev. **7**, 137] that if the nth iterate τ^n of a homeomorphism τ of a compact connected orientable surface φ (which may or may not have boundary curves) upon itself is homotopic to the identity, then τ is homotopic to a homeomorphism whose nth iterate is precisely the identity. Using a fixed point theorem of P. Smith [Ann. of Math. (2) **35**, 572–578 (1934)] the author sketches a short proof of this theorem. [A detailed proof will appear in a projected book by Nielsen and the author, Topology of Surfaces and Their Transformations, Princeton University Press.] The new proof proves a somewhat more general theorem about the extensions of a discontinuous group of motions, and this is shown to be a special case of a problem which is unsolved in its general form. For the case of a torus φ the proof of the theorem is much simpler and may be generalized to obtain a proof of the corresponding theorem for an N-dimensional torus.

R. H. Fox (Princeton, N. J.).

Shizuma, Ryoji. Homotopy properties of fibre bundles. J. Math. Soc. Japan 1, 219–225 (1950).

The first part of this paper starts with the definition of the obstructions to defining a cross section of a fibre bundle in which the base space is a simplicial complex, and the fundamental group of the fiber (which is assumed to be connected) operates trivially on the homotopy groups of the fibre. There then follows an exposition of some of the basic properties of these obstructions. The second part of the paper gives some applications of this theory of obstructions to homogeneous spaces. Perhaps the most interesting of these applications is the following theorem: Let G be a compact, connected, semi-simple Lie group, and T a toral subgroup of G. Then the factor space G/T cannot be homeomorphic to an n-dimensional sphere with $n\neq 2$ (simple examples show that G/T can be a 2-sphere).

W. S. Massey (Providence, R. I.).

Kudo, Tatsuji. Homotopy groups of fibre bundles. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. 1, 56–64 (1950).

Let B be a fiber bundle with base space X, fiber Y, total space Z, group $\mathfrak{A}$, and projection $p:Z\rightarrow X$. The bundle B is called a topological fiber bundle if $\mathfrak{A}$ is the group of all homeomorphisms of Y. In a previous paper [Osaka Math. J. 1, 156–165 (1949); these Rev. 11, 378] the author studied the classification of topological fiber bundles. In the present paper, his main concern is to obtain relations between the homotopy groups $\pi_n(X)$, $\pi_n(Y)$, and $\pi_n(Z)$ for topological fiber bundles. The main result asserts that there exists a sequence of homomorphisms $\alpha_n:\pi_n(X)\rightarrow\pi_{n-1}(\mathfrak{A})$ $(n\geqq 2)$ which are invariants of the bundle structure. Let $p_n:\pi_n(Z, Y)\rightarrow\pi_n(X)$ $(n\geqq 2)$ denote the homomorphisms induced by the projection, and $\partial_n:\pi_n(Z, Y)\rightarrow\pi_{n-1}(Y)$ $(n\geqq 2)$ the homotopy boundary operator. The author defines a homomorphism $\kappa_n:\pi_n(\mathfrak{A})\rightarrow\pi_n(Y)$ $(n\geqq 1)$ and proves that the commutativity relation $\partial_n=\kappa_{n-1}\alpha_n p_n$ holds. The desired relations are now obtained by making use of the exactness of the homotopy sequence of the pair (Z, Y) and the fact that p_n is an isomorphism onto. The methods used to define α_n and prove the theorems depend on the author's paper cited above.

W. S. Massey (Providence, R. I.).

Eilenberg, Samuel, and MacLane, Saunders. Cohomology theory of Abelian groups and homotopy theory. I. Proc. Nat. Acad. Sci. U. S. A. 36, 443–447 (1950).

Soit X un espace connexe par arcs dont les groupes d'homotopie $\pi_i(X)$ sont nuls $(i\geqq 1)$ sauf $\pi_m(X)$ qui est égal à un groupe (abélien) donné Π $(m\geqq 2)$. Dans 2 mémoires antérieurs [Ann. of Math. (2) 46, 480–509 (1945); 51, 514–533 (1950); ces Rev. 7, 137; 11, 735], les auteurs ont montré que les groupes d'homologie (de cohomologie) singulière de X sont isomorphes aux groupes d'homologie (de cohomologie) d'un complexe $K(\Pi, m)$ qui est explicitement défini à l'aide du seul groupe Π (et dépend de la valeur de m). Mais ce complexe ne permet pas de calculs effectifs de l'homologie. Les auteurs cherchent des méthodes purement algébriques de calcul. Ils annoncent d'abord un théorème de "suspension": pour toute valeur de l'entier $k\geqq 1$, ils définissent un homomorphisme des groupes de cohomologie (à coefficients dans un G quelconque):

$$S_k: H^{m+k}(K(\Pi, m+1); G)\rightarrow H^{m+k-1}(K(\Pi, m); G),$$

et prouvent que, pour $k\leqq m$, c'est un isomorphisme sur; pour $k=m+1$, c'est un homomorphisme sur. Donc, pour k donné $\geqq 1$, les $H^{m+k-1}(K(\Pi, m); G)$, quand m varie $\geqq k$, sont isomorphes à un même groupe $Q^k(\Pi, G)$; ce dernier peut être défini directement comme le k-ième groupe de cohomologie d'un certain complexe qui est explicitement décrit. Si Π est une somme directe $\Pi_1+\Pi_2$, $Q^k(\Pi, G)$ est somme directe de $Q^k(\Pi_1, G)$ et $Q^k(\Pi_2, G)$.

Une étude plus poussée de ce complexe et de certains sous-complexes conduit à des groupes de cohomologie $Q^{k,m-1}(\Pi, G)$ dépendant d'un second entier $m\geqq 2$, et qu'on démontre être isomorphes aux $H^{m+k-1}(K(\Pi, m); G)$. Les auteurs annoncent, pour un note ultérieure, une autre définition des groupes $Q^{k,m-1}(\Pi, G)$ qui permettra des calculs effectifs.

H. Cartan (Paris).

Pontryagin, L. S. Characteristic cycles on differentiable manifolds. Amer. Math. Soc. Translation no. 32, 72 pp. (1950).

Translated from Mat. Sbornik N.S. 21(63), 233–284 (1947); these Rev. 9, 243.

GEOMETRY

Emch, Arnold. Rare problems in plane geometry. Scripta Math. 16, 61–66 (1950).

Fog, David. Remarks in connection with a theorem of plane geometry. Mat. Tidsskr. B. 1950, 27–32 (1950). (Danish)

Over the side a_ia_{i+1} of a plane triangle $a_1a_2a_3$ erect (outward) an isosceles triangle with vertex q_{i+2} and base angle u_{i+2} such that $u_1+u_2+u_3=\frac{1}{2}\pi$ (subscripts are to be reduced mod 3). Then $q_1q_2 \sec u_3=q_2q_3 \sec u_1=q_3q_1 \sec u_2$. (For $u_i=\frac{1}{6}\pi$ this becomes the well-known result that $q_1q_2q_3$ is equilateral.) Noticing that the sum of the angles at the a_i in the hexagon $a_1q_3a_2q_1a_3q_2$ is 2π and that $u_i=\frac{1}{2}\pi-\varphi_i$, where $2\varphi_i=\angle a_{i-1}q_ia_{i+1}$, the following fact holds both in the plane and on the sphere. If over the side a_ia_{i+1} an isosceles triangle $a_ia_{i+1}q_{i+2}$ is erected such that the sum of the angles at the a_i in the above hexagon equals 2π, then the angle at q_i in $q_1q_2q_3$ equals half the angle at q_i in $a_{i-1}q_ia_{i+1}$.

H. Busemann (Los Angeles, Calif.).

Gougenheim, André. Un nouveau mode d'accès à la trigonométrie sphérique. C. R. Acad. Sci. Paris 231, 1415–1417 (1950).

Labra y Fernández, Manuel. Discussion of an interesting configuration of Ceva. Revista Soc. Cubana Ci. Fís. Mat. 2, 107–123 (1949). (Spanish)

Coxeter, H. S. M. Self-dual configurations and regular graphs. Bull. Amer. Math. Soc. 56, 413–455 (1950).

A self-dual configuration m_n in projective geometry is a set of m points and m lines, with n of the points (lines) on each line (point). The author describes and discusses some self-dual configurations. In addition to 13 well-known ones he describes a new 12_3 in the real plane. In the course of a discussion of Kummer's 16_6 he describes a new regular skew icosahedron in 6 dimensions. In any graph any ordered sequence of s edges forming a continuous path from one vertex to another in a definite direction is called an s-arc. A graph is s-regular if its group is transitive on the s-arcs

but not on the $(s+1)$-arcs. The author discusses a number of s-regular graphs, with special emphasis on the Levi graphs of the configurations. Such a Levi graph has one "blue" vertex for each point and one "red" vertex for each line. A red and a blue vertex are joined if the corresponding geometrical elements are incident. Two vertices of the same colour are never joined. Maps formed by the embedding of symmetrical graphs in surfaces are also discussed. A map is called regular if its group includes the cyclic permutations of the edges of any face and of the edges meeting at any vertex. In particular the author discusses two infinite families of maps on the torus (maps of quadrilaterals and maps of hexagons). He deduces the theorem that there are infinitely many 2-regular graphs in which just three edges meet at each vertex. *W. T. Tutte* (Toronto, Ont.).

***Michael, W. Ortskurvengeometrie in der komplexen Zahlenebene.** Verlag Birkhäuser, Basel, 1950. 95 pp. 11.50 Swiss francs.

The professed aim of the author is to make available a coherent mathematical presentation of certain curves which are commonly of concern in electrical theory. These are primarily curves whose equations may be expressed in the form $V=R(t)$, where $R(t)$ is a rational function of the real parameter t but with coefficients which are in general complex, and V is the generic point of the curve in the complex plane. The curves discussed from this point of view include the line, the circle, the conics, the circular cubic, and the bicircular quartic. The properties of these curves are developed in some detail, with particular emphasis on obtaining from the parametric form geometric constructions for asymptotes, tangents, osculating circles, and double points. These constructions would be of value in making accurate drawings of the desired curves, and they constitute the chief merit of the book.

It was clearly intended that the work should be available to students without extensive mathematical training, a knowledge of projective geometry and a slight amount of calculus being the only prerequisites beyond ordinary analytic geometry. Unfortunately, however, the mathematical organization and exposition are poor. Ambiguous definitions and hypotheses not clearly stated make certain theorems into dangerous pitfalls for the uncritical reader, and several arguments are unsound. In fact, defects of mathematical exposition are so serious that the reviewer would be unwilling to place the book in the hands of the mathematically immature students for whom it seems to have been written, even though much of the material on graphing curves could well be useful. *S. B. Jackson* (College Park, Md.).

Yaglom, I. M. The Cayley-Klein metrics in the projective plane and complex numbers. Trudy Sem. Vektor. Tenzor. Analizu 7, 276–318 (1949). (Russian)

The nine Cayley-Klein projective metrics defined by real, imaginary, degenerate, or nondegenerate conics are treated in a lucid and uniform manner by the use of three systems of complex numbers: the usual $z=x+iy$; the dual numbers $z=x+\epsilon y$, $\epsilon^2=0$; and the elliptic complex numbers $z=x+ey$, $e^2=1$. There are no essentially new results except in the discussion of Poincaré models. After the coordination of the various geometries with their respective complex domains, the following topics are discussed: the representation of motion by means of linear fractional transformations; the equations of the cycles; equations for points and lines; Poincaré models; conformal and equilong mappings; Möbius and Laguerre transformations of the hyperbolic plane. *H. Busemann* (Los Angeles, Calif.).

Kuiper, N. H. On linear families of involutions. Amer. J. Math. 72, 425–441 (1950).

A maximal linear family of involutions $L(2r+1, 2^r)$ is the set of all those collineations of a complex projective n-dimensional space P_n $(n=2^r-1)$ given by the matrices $X^\alpha\gamma_\alpha$ (α summed from 1 to $2r+1$), where the X^α are arbitrary complex numbers and the γ_α are 2^r by 2^r matrices required to satisfy the identity $(X^\alpha\gamma_\alpha)^2=(\sum_\alpha X^\alpha X^\alpha)\cdot 1_{n+1}$, or, equivalently, $\gamma_\alpha\gamma_\beta=-\gamma_\beta\gamma_\alpha$ for $\alpha\neq\beta$ and $(\gamma_\alpha)^2=1$. The projective space R_m $(m=2r)$ in which the X^α are homogeneous coordinates is called motion space and the P_n in which the involutions operate is called spin space. For a given fixed point ψ in spin space, the images of ψ under the various $X^\alpha\gamma_\alpha$ constitute all the points of a linear subspace of spin space, say $\Xi\psi=\{X^\alpha\gamma_\alpha\psi;$ all values of $X^\alpha\}$. This paper studies the operation Ξ in spin space using occasionally a canonical form for the γ_α but mainly relying on synthetic methods. [Alternatively, one can regard $X^\alpha\rightarrow X^\alpha\gamma_\alpha\psi$ as a linear mapping of motion space into spin space. Then $\Xi\psi$ is the space onto which (the whole of) motion space is mapped.]

For $r=1$, $L(3, 2)$ contains all involutions of the complex projective line, $\Xi\psi$ is the whole line P_1, the involution which "annuls" ψ is the image of ψ under a classical mapping of P_1 onto a conic in a projective plane and the pencil of those involutions which transform ψ into ψ is the tangent line to this conic. It is incorrectly stated in the last paragraph on p. 438 that Ξ coincides with the (unique) null polarity in P_1; this is false since the latter is the identity on points and interchanges the empty set, P_{-1}, and P_1.

For $r=2$, $L(5, 4)$ is the family of involutions obtained if a fixed null polarity in P_3 is multiplied by all the null polarities commutative with it. The operation Ξ is the fixed null polarity. The involutions which carry ψ into P_{-1} correspond to points of a line on a fixed quadric Q_3 in motion space R_4 and those which carry ψ into ψ to the tangent plane containing the line; this geometry also follows from the Plücker-Klein correspondence, which is, however, not used explicitly.

Especially rich in geometric results is the case $r=3$. All the involutions of $L(7, 8)$ commute with just one polarity in P_7 and this determines an invariant quadric Q_6. For ψ a point not on Q_6, Ξ coincides with polarity in Q_6 and $\Xi\psi=$ the polar P_6. For ψ on Q_6, $\psi\rightarrow\Xi\psi=$ some P_3 which always belongs to one of the two families of P_3's (say an axis^2) lying entirely on Q_6. Extending Ξ to the family F of points, lines, axes^1, and axes^2 on Q_6, it is shown to (1) be of period two, (2) carry an axis^2 into a point, a line into a line, and an axis^1 into an axis^1, and (3) preserve adjacence of elements of F, where a line is adjacent to a point or axis on it and any two elements of F are adjacent if they are adjacent to a third element of F in the sense already defined. If Ξ is adjoined to the permutations of F induced by proper and improper (interchanging axes^1 and axes^2) collineations of P_7 leaving Q_6 invariant, a six fold group is obtained and a connection with É. Cartan's principle of triality [Bull. Sci. Math. (2) 49, 361–374 (1925)] is noted.

In P_{15} the involutions of $L(9, 16)$ all commute with a (unique) polarity in a Q_{14}. If ψ is not on Q_{14}, $\Xi\psi$ is a P_8 containing ψ and intersecting a certain T_{10} (=intersection of quadric cones) in a Q_6. If ψ is on Q_{14} but not T_{10}, $\Xi\psi$ is an

axis² in Q_{14} not containing ψ and intersecting T_{10} in a Q_6; if $\psi \epsilon T_{10}$, $\psi\epsilon(\Xi\psi)=$ some P_4 in T_{10}. The last section develops a suggestion due to Room and relates the group generated by the involutions γ_α and their invariant polarity to the construction of the generalized Kummer configurations. Other results, such as those relating the invariant spaces of the involutions to the various quadrics, could not be listed in detail here. *W. Givens* (Knoxville, Tenn.).

Ulčar, Jože. Elementargeometrische Abbildungen und Begriff der Transformationsgruppe in der Geometrie. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Éd. Spéc. 1, 64 pp. (1950). (Macedonian. German summary)

An elementary introduction of F. Klein's ideas about the role of groups in geometry. The booklet is intended primarily for teachers. *W. Feller* (Princeton, N. J.).

Murnaghan, Francis D. The element of volume of the rotation group. Actas Acad. Ci. Lima 13, 9–15 (1950).

An arbitrary rotation in 3-space can be written in an unambiguous manner as the product $R_z(\alpha)R_y(\beta)R_z(\gamma)$ of rotations R_x, R_y, R_z around the x-, y-, z-axis through angles α, β, γ. Using this factorization, the author gives the invariant volume element of the rotation group the expression $\sin\beta\, d\alpha\, d\beta\, d\gamma$. An analogous formula can be given in four dimensions. *H. Freudenthal* (Utrecht).

Murnaghan, Francis D. The element of volume of the rotation group. Proc. Nat. Acad. Sci. U. S. A. 36, 670–672 (1950).

Expansion of the argument in the paper reviewed above to the case of n dimensions. Introduction of Eulerian parameters in the n-dimensional rotation group. Formula for the volume element. *H. Freudenthal* (Utrecht).

Convex Domains, Extremal Problems

Bang, Thøger. On covering by parallel-strips. Mat. Tidsskr. B. 1950, 49–53 (1950).

The author solves the plank problem proposed by Tarski [Parametr 2, 310–314 (1932)]. Let there be given in a finite dimensional Euclidean space a convex body of width Δ which is covered by finitely many parallel strips or planks S_k, the width of S_k being δ_k; then $\sum\delta_k \geqq \Delta$. The proof is based on the following two lemmas. (1) Given a convex body K of width Δ and a vector $2v$ of length $\delta<\Delta$; then the intersection of the two translated bodies $K-\epsilon v$ ($\epsilon=\pm1$) is a convex body of width $\geqq\Delta-\delta$. (2) Given n closed strips S_k, let $2v_k$ be a vector perpendicular to S_k whose length is the width δ_k of S_k. The two closed half spaces whose interiors exactly cover the complement of S_k may be labeled $H_k^{\epsilon_k}$ with $\epsilon_k=+1$ or -1 according as v_k points into or out of the half space. There are 2^n sequences $\epsilon=(\epsilon_1,\cdots,\epsilon_n)$. To each such sequence ϵ is associated the vector $\epsilon v=\sum\epsilon_k v_k$ and the closed polyhedron P_ϵ formed by intersection of the n half spaces $H_k^{\epsilon_k}$. Thus the interiors of the 2^n polyhedra P_ϵ exactly cover the complement of the union of the n strips S_k. The lemma asserts that the 2^n translated polyhedra $P_\epsilon-\epsilon v$ cover the entire space. The theorem in contraposition is easily proved from these two lemmas. Let a convex body K and n strips S_k be given such that $\sum\delta_k<\Delta$: to show that the strips do not cover K. To this end translate that part of K lying in the polyhedron P_ϵ by amount $-\epsilon v$. According to (2) these 2^n translated parts $(K-\epsilon v)\cap(P_\epsilon-\epsilon v)$ cover the intersection, call it K_n, of the 2^n translated bodies $K-\epsilon v$. Now it follows by n successive applications of (1) that K_n is a convex body of width $\geqq\Delta-\sum\delta_k>0$, and hence has interior points. Therefore the interior of some polyhedron P_ϵ must intersect K: whereupon the strips S_k do not cover K. *W. Gustin* (Princeton, N. J.).

Fejes Tóth, László. Some packing and covering theorems. Acta Sci. Math. Szeged 12, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 62–67 (1950).

Definitions: $d=$ convex domain; hexagon $=$ convex polygon with 6 or less sides; $h=$ hexagon of smallest area containing d; $H=$ hexagon of largest area contained in d; $\mathfrak{h}=$ given hexagon. The same letter denotes a domain and its area. Theorems: (1) If n congruent d's, say $d_1,\cdots,d_n$, lie in $\mathfrak{h}$ without overlapping, then $n\leqq\mathfrak{h}/h$. (2) Suppose N congruent d's cover $\mathfrak{h}$, and the boundaries of no two of them intersect in more than two points; then $N\geqq\mathfrak{h}/H$. (3) If n circles $c_1,\cdots,c_n$ lie in $\mathfrak{h}$ without overlapping, then

$$n([c_1^\alpha+\cdots+c_n^\alpha]/n)^{1/\alpha}\leqq\pi(12)^{-\frac12}\mathfrak{h},\qquad \alpha\leqq1-(4-(27)^{\frac12}/\pi)^2/24=.77\cdots.$$

(4) If N circles $C_1,\cdots,C_N$ cover $\mathfrak{h}$, then

$$N([C_1^\beta+\cdots+C_N^\beta]/N)^{1/\beta}\geqq2\pi(27)^{-\frac12}\mathfrak{h},\qquad \beta\geqq1+(2+(27)^{\frac12}/\pi)^2/12=2.11\cdots.$$

Proof of (1): Replace d_1 by a maximal convex domain p_1 which has no interior points in common with $d_2,\cdots,d_n$ $[d_1\subset p_1\subset\mathfrak{h}]$. Similarly, construct successively $p_2,\cdots,p_n$. Then each p_i is a polygon in $\mathfrak{h}$ with, say, ν_i sides, and the p_i's do not overlap. From Euler's formula, $\sum_1^n\nu_i/n\leqq6$. Let $a(\nu_i)$ be the smallest polygon containing d_i and with not more than ν_i sides. Thus $\mathfrak{h}\geqq\sum_1^n p_i\geqq\sum_1^n a(\nu_i)$. From a theorem by Dowker [Bull. Amer. Math. Soc. 50, 120–122 (1944); these Rev. 5, 153] $a(\nu)$ is a convex function of ν. Hence, Jensen's inequality yields $\sum_1^n a(\nu_i)\geqq na(\sum\nu_i/n)$. Since $a(\nu)$ is monotonically decreasing, $a(\sum\nu_i/n)\geqq a(6)=h$. *P. Scherk* (Saskatoon, Sask.).

Vincze, Stephen. On a geometrical extremum problem. Acta Sci. Math. Szeged 12, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 136–142 (1950).

This paper deals with the problem of finding, among all convex polygons with a given number n of sides of length 1, those with minimal diameter. In general, the regular polygon is not among the solutions. If n contains at least one odd prime factor, the minimal value of the diameter is found and extremal polygons are obtained as follows: For any odd factor p of n consider the regular Reuleaux-p-gon whose width is the minimal diameter. The n-gon with sides of length 1 inscribed in it in such a way that every vertex of the Reuleaux-polygon is also a vertex of the n-gon is extremal. This shows that, in general, the solution is not unique. If n is a power of 2, only upper and lower estimates for the minimal diameter are found and it is shown by an example that in the case $n=8$ the regular polygon is not extremal. As the author mentions, the main results are contained in a paper by Reinhardt [Jber. Deutsch. Math. Verein. 31, 251–270 (1922)] which came to his knowledge after writing the present paper. *W. Fenchel.*

***Weyl, H. The elementary theory of convex polyhedra.** Contributions to the Theory of Games, pp. 3–18. Annals of Mathematics Studies, no. 24. Princeton University Press, Princeton, N. J., 1950. $3.00.

Translation by H. W. Kuhn of a paper in Comment. Math. Helv. 7, 290–306 (1935).

Hadwiger, H. Beweis der isoperimetrischen Ungleichung für abgeschlossene Punktmengen. Portugaliae Math. 8, 89–93 (1949).

Let V be the volume (measure) and F the Minkowski surface area of a closed and bounded point set in the k-dimensional Euclidean space. Denote the volume of the unit sphere in this space by ω_k. Then the isoperimetric inequality $F^k \geqq \omega_k k^k V^{k-1}$ is valid. Several proofs of this general result have been given, the first one by Lusternik [C. R. (Doklady) Acad. Sci. URSS (N.S.) 8 (1935 III), 55–58], others by Schmidt [Math. Nachr. 1, 81–157 (1948); these Rev. 10, 471] and Dinghas [ibid. 2, 107–113 (1949); these Rev. 11, 386]. The author gives a short new proof using a selection theorem for closed and bounded sets and Steiner symmetrization. *W. Fenchel* (Princeton, N. J.).

Fáry, István. Sur certaines inégalites géométriques. Acta Sci. Math. Szeged 12, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 117–124 (1950).

By means of known results from integral geometry, the author proves the following inequalities. (1) Consider a closed curve in the ordinary space, and let L denote its length, r the radius of its circumscribed sphere, and G its total (absolute) curvature. Then $\pi L \leqq 4rG$. For a plane curve the sharper inequality $L \leqq rG$ is valid. (2) Consider a closed surface, and let A denote its area, r the radius of its circumscribed sphere, and K the total absolute curvature, that is, the integral of the absolute value of the Gauss curvature. Then $\pi A \leqq 4r^2 K$. *W. Fenchel.*

Zalgaller, V. A. The circle on a convex surface. The local almost-isometry of a convex surface to a cone. Mat. Sbornik N.S. 26(68), 401–424 (1950). (Russian)

Denote by $\rho_S(x, y)$ the distance of the points x, y measured on the surface S in E^3. Every point z of a convex surface F has a circular neighborhood $U_R(z)$ (defined by $\rho_F(z, x) < R$) with the following property: There is a topological mapping $x \rightarrow x'$ of $U_R(z)$ as a neighborhood of z on the tangent cone K of F at z such that for every $\epsilon > 0$ there exists a $\delta > 0$ with $|\rho_F(x, y) - \rho_K(x', y')| \leqq \epsilon \rho_F(x, y)$ for $x, y \epsilon U_\delta(z)$. The boundary $C_R(z)$ (consisting of the points x with $\rho_F(z, x) = R$) of $U_R(z)$ is called a circle. A radius of $C_R(z)$ is any shortest connection of z to a point p of $C_R(z)$. The radius from z to p need not be unique, nor is $C_R(z)$ necessarily a simple closed curve. It is shown how, in spite of these difficulties, an analogue to a representation of $C_R(z)$ in terms of the polar angle at z and the length $l(R)$ of $C_R(z)$ in terms of this representation can be defined. The length $l(R)$ is increasing and continuous for small R. Moreover, $l(R)/R$ is nonincreasing for all R and $l(R)/R \leqq \theta$, where $\theta \leqq 2\pi$ is the complete angle of F at z, that is, the angle at the apex of the tangent cone K of F at z after K has been developed in a plane. *H. Busemann.*

Aleksandrov, A. D. Surfaces represented by the differences of convex functions. Doklady Akad. Nauk SSSR (N.S.) 72, 613–616 (1950). (Russian)

Let F be a surface in E^3 which can be represented locally in the form $z = f_1(x, y) - f_2(x, y)$, where f_i is a convex function. The surface F can be approximated by analytic surfaces for which the integral over the absolute value of the Gauss curvature is uniformly bounded. Hence F has bounded curvature in the sense previously defined by the author [same Doklady (N.S.) 63, 349–352 (1948); these Rev. 10, 325], and all results derived for these surfaces hold for F. In addition, a shortest arc on F has at every point right and left hand tangents, and F has at every point p a tangent cone. The intrinsic metric on this cone approximates the metric of F in the vicinity of p. The measure of the spherical image of a domain on F equals the total intrinsic curvature of the domain in the same sense as in the analogous result for convex surfaces established by the author [Intrinsic Geometry of Convex Surfaces, OGIZ, Moscow-Leningrad, 1948; these Rev. 10, 619]. Also, as in the last reference, the intrinsic area of a domain on F equals the extrinsic area. *H. Busemann* (Los Angeles, Calif.).

Viola, Tullio. Su un problema metrico relativo alle superficie quadrabili. Boll. Un. Mat. Ital. (3) 5, 109–120 (1950).

Given a segment s of unit length embedded in a surface S of finite area, where S bounds a domain V in space, the author considers the infimum L^* of the length $L(C)$ of a curve C with the same extremities as s, when C is restricted to lie (except for these extremities) outside the closure of V. An example is given in which $L^* > 1$ and in which further the admissible curves C for which $L(C) \rightarrow L^*$ tend to the segment s. *L. C. Young* (Madison, Wis.).

Algebraic Geometry

★Severi, Francesco. La géométrie algébrique italienne. Sa rigueur, ses méthodes, ses problèmes. Colloque de géométrie algébrique, Liège, 1949, pp. 9–55. Georges Thone, Liège; Masson et Cie., Paris, 1950. 200 Belgian francs; 1400 French francs.

The paper is divided in two parts: the first section is an attempt to establish that the work of the Italian school of geometers has been in some sense rigorous, while the second section contains a list of unsolved problems in algebraic geometry. The author distinguishes two kinds of rigour: "formal" and "substantial" rigour. Although the difference between these two concepts is not too precisely described in the paper, it seems to the reviewer that "substantial rigour" is meant to denote a good general understanding of the mathematical objects under consideration, while "formal rigour" would probably be about what most mathematicians usually call rigour. In order to establish that the Italian geometers did not lack the gift for "substantial" rigour, the author discusses in some detail (among others) the following statement: The set of all V_k^μ in S_r (V_k^μ represents a variety of dimension k and degree μ) constitutes an algebraic variety. He shows that, long before the work of Chow, the Italian geometers had very "imperious" reasons for holding this assertion to be true. Besides the general feeling that whatever is obtained by algebraic operations must be of an algebraic nature, Severi refers to the construction by Bertini of a one-to-one correspondence between the set of those V_k which are not degenerate (in a certain sense) and a subset of an algebraic variety. It seems to the reviewer that a discussion of this example is apt to throw some light on the distinction between "substantial" and "formal" rigour. A "formal rigorist" has no objection against the procedure which consists in taking provisorily for granted the validity of a certain statement and deriving results from it (as, for instance, Hardy investigated the consequences of the truth of the Riemann hypothesis). However, the formal rigorist would insist that the statement he accepts as an axiom should have a precise meaning, a condition which does not

seem to be satisfied by the assertion quoted above. Now it is certainly meaningful to ask whether a subset of a projective space is or is not an algebraic variety; but the set E of all V_k^p is not given as a set of points in a projective space. Were one to interpret the assertion to mean that there exists a one-to-one correspondence between E and an algebraic variety, then it would merely say that the cardinal number of E is the same as that of some algebraic variety, information which would obviously be of very limited interest in algebraic geometry. Obviously, what is looked for is a one-to-one correspondence between E and an algebraic variety which should have some other definite properties besides being one-to-one; a "formal rigorist" would probably demand that such additional properties be explicitly stated before the statement is accepted even in the form of an unproved axiom. It seems therefore to the reviewer that the assertion that the work of the Italian school of geometers has been rigorous can only be accepted in the light of a very radical distinction between the two kinds of rigour mentioned by the author.

Here are some of the unsolved problems mentioned in the second part of the paper: (a) to extend the theorem of the base to algebraic equivalences of subvarieties of arbitrary dimension of a given variety; and also to give purely algebraico-geometric proofs of this theorem in the cases where it has already been established by transcendental methods; (b) to find the exact conditions of validity of the statement concerning the completeness of the characteristic series cut on a curve C by a complete algebraic system containing C; (c) to prove the birational invariance of the arithmetic genus of a variety; (d) to extend the Riemann-Roch theorem to varieties of dimensions >2; (e) to develop further the author's theory of quasi-Abelian functions.

In a footnote on p. 21 of his paper, the author protests against certain allegedly "impertinentes et injustes" criticisms of his work, and more particularly of his 1942 book on equivalence systems, which have been published in these Reviews; he states that he prefers not to name the "jeune monsieur," author of these reviews. Severi's book on equivalence systems has been reviewed by D. Pedoe in these Rev. **10**, 206; the review does not contain any criticism whatsoever, unjust or otherwise. It seems likely that Severi meant to refer to the review of another book of his, "Fondamenti di geometria algebrica" [CEDAM, Padova, 1948] which was also written by D. Pedoe and which was printed on the same page as the one mentioned above; in this review, objections are raised against an alleged tendency on the part of Severi to give detailed references "only . . . when the author can claim that enough work has been done on certain theorems in Italy to make their non-Italian origin comparatively unimportant." Not having read the book in question, the present reviewer has no opinion on the question whether this criticism is just or not. However, the present reviewer has found nothing in any of the published reviews by D. Pedoe of works by Severi which could be called "impertinent," while, on the contrary, the footnote referred to above is certainly impertinent to D. Pedoe.

C. Chevalley (New York, N. Y.).

Severi, Francesco. Legami tra certe proprietà aritmetiche delle superficie e la teoria della base. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **9**, 59–69 (1950).

Let H be a general linear pencil of curves of genus p on a surface F, and let t be a parameter for the pencil. Let $C(t)$ be the general curve of H and let $V(t)$ be the Jacobi variety of $C(t)$. Let I be the set of points of $V(t)$ which are rational in t. The fundamental theorem of the base is equivalent to the following proposition. If F is regular all points of I can be obtained from a finite number of them by the operations of point addition and subtraction defined on $V(t)$, while if F is of irregularity $q>0$, the points of I are distributed among Abelian varieties M_q (each of which is uniquely determined by any one of its points), and the M_q can all be obtained from a finite number of them by the operations of addition and subtraction. If $I=V(t)$, then F is birationally equivalent to a ruled surface of genus p.

H. T. Muhly (Iowa City, Iowa).

Chatelet, F. Sur les points multiples des courbes algébriques planes. Cahiers Rhodaniens **1**, 9 pp. (1949).

The author employs a rational transformation originally defined by G. Dumas [Comment. Math. Helv. **1**, 120–141 (1929)] to analyze the singular points of algebraic plane curves, replacing the nonlinear transformation by a linear correspondence of its exponents. He shows that the Dumas transformation includes as special cases both the development of Puiseux and the transformation used by Noether to analyze singular points. *T. R. Hollcroft* (Aurora, N. Y.).

Piazzolla Beloch, Margherita. Topologia delle curve situate sopra superficie generali del 3° ordine con meno di 27 rette reali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 576–578 (1950).

Real circuits of curves on the real sheet of a cubic surface are classified according to the number of real lines on the surface to which they are parisecant or disparisecant, i.e., which they meet in an even or odd number of points. On surfaces of types II, III (with 15 and 7 real lines, respectively) an even circuit is parisecant and an odd circuit disparisecant either to all or to precisely 7 (respectively 3) of these. A nonsingular algebraic curve of any order cannot have more than 5 (respectively 3) odd circuits, and if it has one disparisecant to all the lines it can have no other odd circuit. There exist curves with this maximum number of odd circuits. On surfaces of types IV, V (with 3 real lines, but respectively 1 and 2 sheets) an even circuit is parisecant and an odd circuit disparisecant to all the lines. A nonsingular algebraic curve has (as in the plane) precisely no or one odd circuit according as its order is even or odd. Corresponding results for type I (with 27 real lines) were obtained by the author in a previous paper [Rend. Circ. Mat. Palermo **55**, 1–20 (1931)]. *P. Du Val.*

✱Châtelet, François. Application des idées de Galois à la géométrie algébrique. Colloque de géométrie algébrique, Liège, 1949, pp. 91–103. Georges Thone, Liège; Masson et Cie., Paris, 1950. 200 Belgian francs; 1400 French francs.

By a neat application of Galois Theory the author deals with the classical problem of the existence of rational points on a unicursal curve. By the example of the rational points on a rational quadric the author shows how the use of algebraic numbers is helpful and natural even in problems which can be stated entirely in terms of the rational field.

D. B. Scott (London).

Permutti, Rodolfo. Sui moduli delle curve k-gonali. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **9**, 54–58 (1950).

An algebraic curve is said to be k-gonal if it contains a g^1_k but not a g^1_{k-1}. Segre has shown [Math. Ann. **100**, 537–

551 (1928)] that the k-gonal curves of genus p depend on $N=2k+2p-5$ moduli. In the present note it is shown that for general moduli such curves cannot contain a g_h^1 with $k<h<\frac{1}{2}p+1$, and those which do form a family depending on $N'=2k+2h+p-7$ moduli. *H. T. Muhly.*

Fano, Gino. Nuove ricerche sulle varietà algebriche a tre dimensioni a curve-sezioni canoniche. Pont. Acad. Sci. Comment. **11**, 635–720 (1947).

In questa lunga memoria l'autore porta a compimento una serie di ricerche, che si estende per lo spazio di circa quarant'anni. Si tratta delle ricerche relative alla razionalità od irrazionalità delle varietà algebriche M_3^{2p-2} a tre dimensioni e d'ordine $2p-2$, immerse nello spazio proiettivo a $p+1$ dimensioni; e quindi a curve-sezioni canoniche di genere p ($\geqq 3$). Sin dal 1908 l'autore ha dimostrato che le varietà corrispondenti ai casi $p=3$ e $p=4$ sono generalmente irrazionali; scoprendo così i primi esempi di varietà non razionali con i generi tutti nulli e mettendo di conseguenza in luce la difficoltà di assegnare le condizioni necessarie e sufficienti per la razionalità delle varietà a tre dimensioni. Successivamente, i principali contributi dell'autore allo studio delle M_3^{2p-2} sono stati i seguenti: le M_3^{2p-2} di S_{p+1} esistono soltanto per $p\leqq 37$ e per $p>10$ sono tutte razionali, all'infuori forse della M_3^{24} di S_{14} ($p=13$), la quale è birazionalmente identica alla M_3^{14} di S_9 ($p=8$) sezione generica della varietà di Grassmann delle rette di S_5 ed alla varietà cubica generale di S_4; le M_3^{2p-2} di S_{p+1}, che contengono solo superficie intersezioni complete con forme, sono razionali nei casi $p=7$, 9, 10. In base a tali risultati restavano soltanto da approfondire i casi $p=5$, 6, 8. Ciò è appunto quanto ha fatto l'autore nella memoria in esame. Lo strumento essenziale usato dall'autore è la determinazione sopra una M_3^{2p-2} del tipo anzidetto di tutti gli eventuali sistemi lineari di superficie di generi uno, completi e segati da forme di ordine $n>1$, i quali conducono a rappresentare la data varietà M sopra un'altra dello stesso tipo, per un valore di p sia diverso che uguale a quello di partenza. Se ora la M_3^{2p-2} ($p=5$, 6, 8) fosse razionale, dovrebbero esistere su di essa sistemi lineari di superficie di generi uno rappresentativi di M_3^{2p-2} corrispondenti ai valori 7, 9, 10, per le quali la razionalità è nota. Poichè si riesce a dimostrare che ciò non succede, si conclude la irrazionalità della M_3^{2p-2} per $p=5$, 6, 8. Occorre tuttavia dire che la dimostrazione dell'impossibilità anzidetta è data sotto l'esplicita ipotesi che la corrispondenza tra una M_3^{2p-2} ed un'altra μ_3^{2p-2}, nella quale al sistema lineare delle sezioni iperpiane della M (o della μ) corrisponde sulla μ (o sulla M) un sistema lineare di superficie con i generi unitari, sia una corrispondenza "regolare" in un senso ben precisato. Solo sotto tale ipotesi, la quale per quanto possa apparire estremamente plausibile potrebbe tuttavia essere a priori restrittiva, rimane acquisita la irrazionalità delle M_3^{2p-2} di S_{p+1} per $p=5$, 6, 8, contenenti soltanto superficie intersezioni complete con forme; e quindi, tenuto conto che la M_3^3 generale di S_4 è birazionalmente equivalente alla M_3^{14} di S_9 ($p=8$) risulta anche provata la non razionalità della M_3^3 generale di S_4.

Non è qui assolutamente possibile riferire sui numerosissimi particolari dimostrativi, sugli importanti risultati collaterali e su tutti gli acuti accorgimenti, che mettono questa memoria tra le più elaborate e profonde, che siano state scritte con i metodi della scuola italiana di geometria algebrica. Occorre tuttavia almeno menzionare le considerazioni dell'autore sulle trasformazioni birazionali in sè delle varietà M_3^{2p-2}. La memoria apre altresì la via, come l'autore stesso afferma, ad ulteriori e più approfondite ricerche. Specialmente importanti appaiono quelle ricerche future, che tenderanno a valutare esattamente la portata della anzidetta ipotesi di regolarità e di qualche altra ipotesi di generalità, che l'autore è costretto talvolta a fare in conseguenza della difficoltà dell'argomento; in ordine soprattutto a stabilire il sussistere dei risultati generali della memoria anche negli eventuali casi, in cui siffatte ipotesi vengano meno. *F. Conforto* (Roma).

Roth, Leonard. Algebraic varieties with canonical curve sections. Ann. Mat. Pura Appl. (4) **29**, 91–97 (1949).

L'auteur poursuit dans le cas d'une dimension générale r, les recherches de Fano sur la rationnalité des variétés à trois dimensions à courbes-sections canoniques générales de genre p, se limitant au cas des variétés dites de première espèce dont les V_3 sections ne contiennent que des intersections complétés par des formes de l'espace. Il montre que si $r\geqq 4$, ces variétés sont rationnelles dès que $p\geqq 7$, si $p=5$ elles sont rationnelles dès que r dépasse ou égale 5, enfin si $p=6$ elles sont rationnelles pour $p=5$. La démonstration s'appuie sur les principaux résultats de Fano [voir l'analyse ci-dessus] et de l'auteur [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 541–545 (1947); ces Rev. **10**, 143] et sur la projection à partir d'une courbe normale non spéciale appartenant à une section permettant de passer de p à $p'<p$; il note également que si une V_{n-r} est section complète de n quadriques et contient simplement un espace linéaire S_{n-1} la variété est rationnelle. (À noter que cette condition de rationnalité est suffisante, mais non nécessaire.) Le traitement de $p=6$ demande une étude spéciale. L'auteur termine en donnant un certain nombre de formes de S_5 rationnelles intéressantes. *B. d'Orgeval* (Grenoble).

Roth, L. On fourfolds with canonical curve sections. Proc. Cambridge Philos. Soc. **46**, 419–428 (1950).

In a recent paper [see the preceding review] the author investigated varieties of the type $V_r^{2p-2}[p+r-2]$, $r>3$, for which the generic prime section is a Fano threefold of the first species, i.e., such that its generic curve section is a canonical curve of general character while its generic surface section contains only complete intersections with primals; he showed that such of these varieties as exist must be rational if $p>6$, $r>3$, and also if $p=5$, $r>4$, or $p=6$, $r=5$. The present work deals in more detail with the series for $r=4$; the space representations of the rational fourfolds of the series are obtained, and the author reaches the conclusion that the series terminates at $p=10$. A V_4^{2p-2} of the type considered contains an irreducible system of ∞^3 lines of known or computable numerical characters, and also an ∞^5 system of conics, and it projects from one of its lines or conics into another fourfold with canonical curve sections; but this, since it contains a planar V_3^3 or V_3^4 arising from the vertex of projection, is not of the first species. Most of the representations obtained are direct projections and depend on the results just stated.

The cases considered are as follows. The general V_4^8 of the series, the intersection of three general quadrics in [7], is presumed to be irrational, though it is certainly unirational: but the special V_4^8 which contains a plane π projects birationally from π and is the projective model of quartic primals of S_4 through a surface ${}^3F^9$ of arithmetic genus 3. The general V_4^{12} of the series projects (birationally)

from a tangent [4] onto S_4 and is mapped by septimic primals passing doubly through a rational ${}^6F^8$ and simply through 5 skew planes which meet ${}^6F^8$ in nonsingular quartic curves. The surface ${}^6F^8$ is the model of a system of plane curves $C^6[2^4, 1^{12}]$. If the V_4^{13} contains (exceptionally) a rational normal cubic scroll, it projects birationally from the latter, and the resulting representation on S_4 is by quartic primals through a (general) projected Bordiga ${}^3F^9$. The V_4^{14} of the series, general [10]-section of the Grassmannian of lines in [5], can be mapped on S_4 by quartic primals through a rational surface ${}^4F^7$, model of a system $C^6[2^6, 1^5]$. The general V_4^{16} of the series admits two birational projections on S_4. The first, from a line and a tangent [4], leads to a representation of order 9. The second, from a line and then from a cubic scroll of the resulting V_4^{13}, gives a representation on S_4 by septimic primals passing doubly through a singular (projected) Bordiga ${}^3F^9$ and simply through five planes. The ${}^3F^9$ has a mixed 4-ple point O composed of a proper triple point at O and a simple branch surface passing through O, and the five planes meet ${}^3F^9$ in quartic curves which pass twice through O on the cubic nodal cone and once on the simple branch. If V_4^{16} contains a Del Pezzo F^5, it can be mapped on S_4 by quartic primals through a projected Del Pezzo F^8. The V_4^{18} of the series projects birationally from a conic and a tangent [4], the resulting representation being of order 9. In this last case only, no argument for the existence of the type of variety in question is suggested. *J. G. Semple.*

Jongmans, François. Remarques sur les formes qui contiennent une variété algébrique donnée. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 476–479 (1950).

Si l'on se donne t variétés $V_{r-2}^{(i)}$ d'ordre n, irréductibles ou non, situés dans t hyperplans $\Sigma^{(i)}$ distincts dont deux ont toujours une section hyperplanes totale V_{r-3} commune et dont trois n'ont jamais une section hyperplane ou un espace linéaire S_{r-3}, en commun, il existe dans S_r un système linéaire complet irréductible ∞^k, $k=\binom{n-t+r}{r}$, admettant comme sections hyperplanes les variétés $V_{r-2}^{(i)}$. Ceci se montre par récurrence. Le résultat précédent permet de libérer les résultats obtenus par Roth [mêmes Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 541–545 (1947); ces Rev. **10**, 143] des hypothèses restrictives r impair ou $n\leqq r+1$.

B. d'Orgeval (Grenoble).

Scott, D. B. The united curve of a point-curve correspondence on an algebraic surface, and some related topological characters of the surface. Proc. London Math. Soc. (2) **51**, 308–324 (1950).

In an earlier paper [Proc. Cambridge Philos. Soc. **36**, 414–423 (1940); these Rev. **2**, 137] the author introduced a topologically and birationally invariant module $\mathfrak{J}$ associated with an irreducible algebraic surface F, where $\dim \mathfrak{J}=t$ is at most equal to the dimension ρ of the module $\mathfrak{A}$ of algebraic curves on F. Here he shows that if $t<\rho$ there exist two topologically invariant sub-modules $\mathfrak{Q}$ and $\mathfrak{D}$ of $\mathfrak{A}$, of dimensions t and $\rho-t$, respectively. Let $C_1, \cdots, C_\rho$ be a basis for the algebraic curves on F such that for the associated matrices [see the review of the paper cited above] $\underline{a}^{(1)}, \cdots, \underline{a}^{(\rho)}$ we have $\underline{a}^{(t+1)}=\cdots=\underline{a}^{(\rho)}=0$. Then $C_{t+1}, \cdots, C_\rho$ form a basis for $\mathfrak{D}$. Let $\underline{y}$ be the inverse matrix of $\|C_i\cdot C_j\|$ and $\mathfrak{Q}'$ the module with basis $\sum_{j=1}^{\rho} y_{ij}C_j$, $i=1, \cdots, t$; $\mathfrak{Q}$ is the module of integral curves in $\mathfrak{Q}'$. The dimension of the common part of $\mathfrak{Q}$ and $\mathfrak{D}$ is called the "overlap" of F, and is a birational invariant. Examples are given to show that the overlap may or may not be zero; in particular, a ruled surface has overlap 1. Finally, a new connection is made with the author's work on point-curve correspondences [ibid. **41**, 135–145 (1945); these Rev. **7**, 27]. If U is the united curve of such a transformation, and X and Y the transform and inverse transform of a general point, then $U\approx X+Y+Q$, where $Q\epsilon\mathfrak{Q}$. The paper closes with some speculations on overlapped surfaces and on the possibility of $\mathfrak{Q}$ consisting of multiples of a canonical curve in the case $t=1$. *R. J. Walker* (Ithaca, N. Y.).

Benedicty, Mario. Equazione canonica della generica trilinearità piana di dimensione cinque. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **9**, 192–204 (1950).

With a trilinear correspondence, $\sum_1^3 a_{ijk}x_iy_jz_k=0$ among three planes Π_x, Π_y, Π_z, there are associated three cubics C_x, C_y, C_z, given by $\det \alpha_{jk}=0$, $\det \beta_{ik}=0$, $\det \gamma_{ij}=0$, where $\alpha_{jk}=\sum_i a_{ijk}x_i$, $\beta_{ik}=\sum_j a_{ijk}y_j$, $\gamma_{ij}=\sum_k a_{ijk}z_k$. These cubics are birationally equivalent. It is shown that there exist homogeneous coordinate systems in the planes Π_x, Π_y, Π_z in which the trilinear correspondence takes the form

$$\rho(x_1y_1z_1+x_2y_2z_2+x_3y_3z_3)+\sigma(x_1y_2z_3+x_2y_3z_1+x_3y_1z_2)+\tau(x_1y_3z_2+x_2y_1z_3+x_3y_2z_1)=0,$$

provided a_{ijk} is general. In such systems C_x, C_y, C_z have the form $\xi_1^3+\xi_2^3+\xi_3^3-(\rho^3+\sigma^3+\tau^3)(\rho\sigma\tau)^{-1}\xi_1\xi_2\xi_3=0$. Projective properties of the correspondence are deduced from this canonical form. *H. T. Muhly* (Iowa City, Iowa).

Defrise, Pierre. Étude locale des correspondances rationnelles entre surfaces algébriques. Mém. Soc. Roy. Sci. Liège (4) **9**, no. 3, iii+133 pp. (1949).

An expository article in which the Enriques theory of infinitely near points is applied to the study of local properties of rational correspondences between algebraic surfaces defined over the complex field. *H. T. Muhly.*

Differential Geometry

Bouligand, Georges. Concomitance et asymptotiques généralisées d'une surface. C. R. Acad. Sci. Paris **231**, 1194–1195 (1950).

Fabricius-Bjerre, Fr. On evolutes of a circle and analogous space curves. Mat. Tidsskr. B. **1950**, 10–15 (1950). (Danish)

A plane curve is an evolute of a circle with centre O if and only if the square of the distance of a variable curve point from O is a linear function of the arc length. The author investigates the space curves satisfying the same condition. Such a curve is characterized by the following property: Unfold the cone projecting the curve from O into the plane. Then the curve goes over into an evolute of a circle. Various properties of these curves are proved. Example: A space curve belongs to the class considered if and only if it is an orthogonal trajectory to a family of tangent planes of a sphere. Several generalizations are mentioned. *W. Fenchel* (Princeton, N. J.).

Hsiung, Chuan-Chih. A note of correction. Proc. Amer. Math. Soc. **1**, 824–825 (1950).

Corrections to a paper which appeared in Bull. Amer. Math. Soc. **55**, 623–628 (1949); these Rev. **11**, 132.

Hartman, Philip, and Wintner, Aurel. On the fundamental equations of differential geometry. Amer. J. Math. **72**, 757–774 (1950).

Every surface of class C^2 in Euclidean 3-space possesses first and second fundamental forms whose matrices $g_{ij}(u, v)$, $h_{ij}(u, v)$ are of class C^1 and C^0, respectively. However, the classical existence theorem of Bonnet assumes that the matrices g_{ij}, h_{ij} are of class C^2 and C^1, respectively, and, subject to well-known additional conditions, establishes the existence of a surface of class C^3 for which g_{ij} and h_{ij} are the matrices of the first and second fundamental forms. The authors bridge the gap between these two results. The main theorem states that necessary and sufficient conditions that symmetric matrices g_{ij}, h_{ij}, defined in a simply connected region R, be the matrices of the first and second forms of a surface of class C^2 are that their classes be C^1 and C^0, respectively, g_{ij} be positive definite, and an integrated form of the Gauss-Codazzi equations be satisfied for arbitrary closed curves of class C^1 in R. An analogous theorem is proved concerning the existence theorem for curves based on the Frenet equations. A number of results dealing with the various definitions of a torse and their equivalence for surfaces of class C^2 are also given. *A. Fialkow.*

Hartman, Philip. On the local uniqueness of geodesics. Amer. J. Math. **72**, 723–730 (1950).

It is shown that on a surface of class C^2 in Euclidean 3-space, every geodesic is uniquely determined (locally) by its initial conditions. Indeed, by a transformation of class C^1 it is possible to introduce either geodesic polar coordinates or geodesic parallel coordinates at any point of a surface of class C^2. However, a counter-example is used to show that the geodesic may not be uniquely determined if the surface is only of class C^1. The proofs depend upon the use of the theorem of Gauss-Bonnet and an integrated form of the Frobenius formula for the Gaussian curvature.

A. Fialkow (Brooklyn, N. Y.).

Hoesli, Rudolf J. Spezielle Flächen mit Flachpunkten und ihre lokale Verbiegbarkeit. Compositio Math. **8**, 113–141 (1950).

In a neighborhood of $x=y=0$ let an analytic surface F be represented in the form $z=\varphi^{(n)}(x, y)+\varphi^{(n+1)}(x, y)+\cdots$, where $\varphi^{(j)}(x, y)$ is a form of degree j, $\varphi^{(n)}(x, y)\not\equiv 0$, and $n\geqq 1$. If $n>1$, the origin is a flat point of F. Let

$$K(x, y)=K^{(\kappa)}(x, y)+K^{(\kappa+1)}(x, y)+\cdots,$$
$$0\leqq\kappa<\infty, \quad K^{(\kappa)}(x, y)\not\equiv 0,$$

represent the Gauss curvature of F (so that $K\equiv 0$ is excluded). Then κ is called the order of the curvature at the origin. A given analytic line element has locally always analytic realizations without flat points. However, there are line elements which do not have realizations for which a given point is flat, even if the order of the curvature at this point is arbitrarily large (but $\kappa<\infty$). There are line elements which have in the neighborhood of a given point exactly one realization (up to motions) which is flat at that point, and even line elements of the last type for which this realization is rigid, that is, does not belong to a continuous family of intrinsically isometric but not congruent surfaces. The paper is a slight generalization and a considerable simplification of a paper by Efimoff [Rec. Math. [Mat. Sbornik] N.S. **19**(61), 461–488 (1946); these Rev. **8**, 338]. *H. Busemann* (Los Angeles, Calif.).

Yanenko, N. N. The structure of deformable surfaces in a many-dimensional Euclidean space. Doklady Akad. Nauk SSSR (N.S.) **72**, 857–859 (1950). (Russian)

Yanenko, N. N. On some projectively invariant properties of deformable surfaces in a many-dimensional Euclidean space. Doklady Akad. Nauk SSSR (N.S.) **72**, 1025–1028 (1950). (Russian)

In E^{m+q} consider two m-dimensional manifolds V_m and $\bar{V}_m$ of class C^{III}. If a one-to-one mapping of V_m on $\bar{V}_m$ exists such that at corresponding points the line elements of V_m and $\bar{V}_m$ are equal, then V_m and $\bar{V}_m$ are called isometric and we write $V_m\sim\bar{V}_m$. The isometry is called proper if no manifolds $V_{m+s}\supset V_m$ and $\bar{V}_{m+s}\supset\bar{V}_m$, $0<s<q$, exist such that $V_{m+s}\sim\bar{V}_{m+s}$ induces $V_m\sim\bar{V}_m$. If $I_1, \cdots, I_{m+q}$ are the unit vectors of a properly closed rectangular coordinate system (attached to a variable point of V_m) such that $I_1, \cdots, I_{m+q}$ are tangent to V_m and $J_1, \cdots, J_{m+q}$ are defined analogously for $\bar{V}_m$, then the Cartan forms of V_m and $\bar{V}_m$ defined by $dr=\sum_{\alpha=1}^{m+q}\omega^\alpha I_\alpha$, $d\bar{r}=\sum_{\alpha=1}^{m+q}\Omega^\alpha J_\alpha$, $dI_\alpha=\sum_{\beta=1}^{m+q}\omega_\alpha{}^\beta I_\beta$, $dJ_\alpha=\sum_{\beta=1}^{m+q}\Omega_\alpha{}^\beta J_\beta$, $\alpha=1, \cdots, m+q$, will satisfy the relations $\omega^\alpha=\Omega^\alpha$, $\alpha=1, \cdots, m$; $\omega^{m+s}=\Omega^{m+s}=0$, $s=1, \cdots, q$; $\omega_j{}^i=\Omega_j{}^i$, $i, j=1, \cdots, m$; $\sum_{s=1}^q[\omega_i{}^{m+s}\omega_j{}^{m+s}]=\sum_{s=1}^q[\Omega_i{}^{m+s}\Omega_j{}^{m+s}]$, $i, j=1, \cdots, m$. The rank of the system of forms $\omega_i{}^{m+s}$, $\Omega_j{}^{m+t}$, $s, t=1, \cdots, q$; $i, j=1, \cdots, m$ is called the rank r of the isometry $V_m\sim\bar{V}_m$. The main result of the first paper is that $r>2q$ implies that the isometry is not proper. The position of $I_1, \cdots, I_m$ and $J_1, \cdots, J_m$ in the tangent planes of V_m and $\bar{V}_m$ can be further specialized so that the relations $\omega_\alpha{}^{m+s}=\Omega_\alpha{}^{m+s}=0$, $s=1, \cdots, q$; $\alpha=r+1, \cdots, m$ and $\omega_i{}^{m+s}=\sum_{j=1}^r\lambda^s{}_{ij}\omega^j$, $\Omega_i{}^{m+s}=\sum_{j=1}^r\mu^s{}_{ij}\omega^j$, $\mu^s{}_{ij}=\mu^s{}_{ji}$ hold. It then follows that relations of the form $\omega_\alpha{}^i=\sum_{j=1}^r a^i{}_{\alpha j}\omega^j$ hold and that the $\lambda^s{}_{ij}$, $\mu^s{}_{ij}$ satisfy (for every s) the equations (*) $\sum_{i=1}^r(a^i{}_{\alpha j}x_{ik}-a^i{}_{\alpha k}x_{ij})=0$, $x_{ij}=x_{ji}$, $\alpha=r+1, \cdots, m$; $i, j, k=1, \cdots, r$. The rank of (*) as equations for x_{ij}, $i\leqq j$, is denoted by R. The main result of the second paper is that if $V_m\sim\bar{V}_m$ has rank r and is proper, then $R\leqq\frac{1}{2}r(r-1)$.

H. Busemann (Los Angeles, Calif.).

Blank, Ya. P. Solution of a problem of Engel on surfaces of translation. Zapiski Naučno-Issled. Inst. Mat. Meh. Har′kov. Mat. Obšč. (4) **19**, 121–140 (1948). (Russian)

This is the third paper in which the author deals with the projective problem of surfaces of translation [Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) **16**, 45–61 (1940); **17**, 99–107 (1940); these Rev. **3**, 17]. In this paper he removes some of the restrictions previously imposed and reduces the problem to the solution of ordinary differential equations of as high as the tenth order. The author gives the various possible solutions. This enumeration also verifies the theorem of Lie that there exists no minimal surface of translation with respect to two planes. *M. S. Knebelman* (Pullman, Wash.).

Gordevskiĭ, D. Z. Affine-parallel surfaces. Zapiski Naučno-Issled. Inst. Mat. Meh. Har′kov. Mat. Obšč. (4) **19**, 141–150 (1948). (Russian)

An affine-parallel surface to $x(u, v)$ is given by $\bar{x}=x+\sigma y$, where y is the affine normal to x and σ is a constant. By investigating the fundamental invariants of the two surfaces, the author shows that if the two surfaces are to have the same affine normals at corresponding points, the two curvatures of $x(u, v)$ must be constant; that is, the surface must be an affine sphere or a linear surface. Other known results are also obtained but, as the author points out, they were previously obtained by less general methods. His own method is quite laborious. *M. S. Knebelman.*

Fáry, István. Quelques remarques sur la définition des espaces de Riemann. C. R. Acad. Sci. Paris **231**, 1410–1412 (1950).

The author's summary is as follows: "Nous montrerons qu'il est possible de définir les espaces de Riemann en associant à chaque point a, un système de coordonnées normales d'origine a et en caractérisant bien le noyau d'espace euclidien défini par ce système de coordonnées. Cette caractérisation est locale et non infinitésimale en ce sens qu'elle ne fait pas intervenir l'espace tangent."

S. Chern (Chicago, Ill.).

Springer, C. E. Union curves of a hypersurface. Canadian J. Math. **2**, 457–460 (1950).

In a previous paper [Bull. Amer. Math. Soc. **51**, 686–691 (1945); these Rev. **7**, 172], the author obtained the differential equations of union curves on a surface in Euclidean 3-space and also defined the union curvature vector of a curve. In the paper under review, these results are generalized to apply to curves on a hypersurface of an n-dimensional Riemann space. *A. Fialkow* (Brooklyn, N. Y.).

Ryžkov, V. V. On metric deformations of different orders. Uspehi Matem. Nauk (N.S.) **5**, no. 4(38), 134–135 (1950). (Russian)

If two subspaces of order n in R_N given by $\bar{x}=\bar{x}(u_1, \cdots, u_n)$ and $\bar{y}=\bar{y}(u_1, \cdots, u_n)$ are to be metrically applicable of order k it is necessary and sufficient that the scalar products of any two vectors of the form $\partial^m\bar{x}/\partial u_1^{\alpha_1} \cdots \partial u_n^{\alpha_n}$, $\sum\alpha_i=m\leqq k$, should have the same values for the two spaces at the points corresponding to any set $u_1, \cdots, u_n$. Since these conditions are not necessarily independent, the author introduces a system of forms $(d^s\bar{x})^2$, $s=1, \cdots, k$, in terms of which the conditions become $\omega_s(u, du)=(d^s\bar{x})^2=(d^s\bar{y})^2$. The forms ω_s define a Riemannian differential geometry of order k and the system ω_s is positive definite if the Gramian matrix of the ω's is positive definite. In this case, $\omega_s(u, du)$ being given, one can find $\bar{x}$ in a space of N dimensions, where $N=\sum_{m=1}^{k}\binom{n+2m-1}{n-1}$ such that $(d^s\bar{x})^2=\omega_s$. If the vectors $\partial^s\bar{x}/\partial u_1^{\alpha_1} \cdots \partial u_n^{\alpha_n}$, $s\leqq 2k$, $\alpha\leqq\min(s, 2k-s)$, are linearly independent, the applicability depends on N_1 functions, where $N_1=\sum_{m=0}^{k-1}\binom{n+2m-1}{n-1}(k-m)/(2m+1)$.

M. S. Knebelman (Pullman, Wash.).

Hlavatý, V. Deformation theory of subspaces in a Riemann space. Proc. Amer. Math. Soc. **1**, 600–617 (1950).

When we effect an infinitesimal deformation

$$^*\xi^\lambda=\xi^\lambda+V^\lambda(x)\epsilon$$

in a Riemann space (or more generally in an affinely connected space), we obtain, for a geometric object $F(\xi)$, three kinds of values at the point $^*\xi$: (1) the value F' of F at $^*\xi$; (2) the value $^\circ F$ obtained by parallel displacement of F from ξ to $^*\xi$ (this case is possible only for a tensor); (3) the value $^*F$ obtained by "dragging" F from ξ to $^*\xi$. Thus we can consider three kinds of operators: (a) $\tau F=\lim_{\epsilon\to 0}[(^*F-{}^\circ F)/\epsilon]$; (b) $\omega F=\lim_{\epsilon\to 0}[(F'-{}^\circ F)/\epsilon]$; (c) $\Delta F=(\tau-\omega)F$. This idea is contained in a paper by Schouten and van Kampen [Prace Mat.-Fiz. **41**, 1–19 (1933)]. The deformation theory of subspaces was developed recently by the reviewer [J. Fac. Sci. Univ. Tokyo. Sect. I. **6**, 1–75 (1949); these Rev. **11**, 688]. The author here studies the deformation theory of a family of subspaces in a Riemann space, the family being such that through any generic point of the space there is only one subspace belonging to the family. He applies successively the three operators mentioned above to the tangent vector fields T_a^λ, to the metric tensor g_{bc}, to the Christoffel symbols Γ^a_{bc}, to the Euler-Schouten tensor $K^\lambda{}_{ab}$, and finally to the generalized Euler-Schouten tensor $K^\lambda{}_{a_r\cdots a_1}$. Then he proves theorems of the following sort: The system of differential equations $\Delta T_a^\lambda=0$ is completely integrable; let $V_1^\lambda, \cdots, V_n^\lambda$ be n linearly independent solutions of $\Delta T_a^\lambda=0$, then $V_b^a(\partial V_c^\lambda/\partial x^a)-V_c^a(\partial V_b^\lambda/\partial x^a)$ is also a solution; a necessary and sufficient condition that the family of subspaces be reproduced (i.e., $\Delta K^\lambda{}_{a_r\cdots a_1}=0$) is that V^λ be a solution of $\Delta T_a^\lambda=0$. Finally he states necessary and sufficient conditions that the family of totally geodesic subspaces will be reproduced, which give generalizations of classical results of Levi-Civita and Jacobi. *K. Yano.*

Tonolo, Angelo. Sopra un sistema di equazioni differenziali relativo ai moti rigidi delle varietà Riemanniane a tre dimensioni a curvatura costante. Rend. Sem. Mat. Univ. Padova **19**, 250–272 (1950).

Let v^ν be the vector of a rigid deformation in a V_3 and w^ν its vortex. Then the Killing equations are equivalent to (1) $\nabla_\mu v_\lambda=\epsilon_{\lambda\mu\nu}w^\nu$ ($\epsilon_{\lambda\mu\nu}$ being the Ricci skew symmetric tensor). Introduce three mutually orthogonal unit vector fields $\underset{a}{i}{}^\nu$ and put $\underset{a}{v}=v_\lambda \underset{a}{i}{}^\lambda$. Then

$$(R_1) \qquad \underset{b}{i}{}^\mu\nabla_\mu \underset{a}{v}=\underset{b}{i}{}^\mu(\nabla_\mu \underset{a}{i}{}^\nu)v_\nu+\underset{a}{i}{}^\omega \underset{b}{i}{}^\mu w^\lambda\epsilon_{\omega\mu\lambda}$$

follows at once from (1). Denote by (R_2) the integrability conditions of (R_1). The author rewrites the sets (R_1), (R_2) (Ricci equations) using Ricci coefficients of rotation in the case of a V_3 of constant curvature K and obtains in particular (2) $\nabla_\mu w_\lambda=Kv^\nu\epsilon_{\lambda\mu\nu}$. [Reviewer's remark. This is a much stronger statement than the one represented by the integrability conditions of (1).] In the remaining part of the paper the author derives some consequences of $(R_1)(R_2)$. Examples: The modul of v^ν along a trajectory is constant and if the trajectories are geodesics, then w^ν and v^ν are linearly dependent. If and only if v^ν is a normal congruence, then v^ν and w^ν are perpendicular. [Reviewer's remark. These three results may be obtained at once from (1).] The author considers also some first integrals of (R_1) and (R_2).

V. Hlavatý (Bloomington, Ind.).

Ruse, H. S. Parallel planes in a Riemannian V_n. Proc. Roy. Soc. Edinburgh. Sect. A. **63**, 78–92 (1950).

Ce papier fait suite aux travaux de A. G. Walker [Quart. J. Math., Oxford Ser. (1) **20**, 135–145 (1949); ces Rev. **11**, 460] et de l'auteur [ibid., 218–234 (1949); ces Rev. **11**, 461] sur les champs parallèles de plans partiellement nuls dans un espace riemannien V_n de signature quelconque. Dans la suite le mot "champ" sera supprimé. Un p-plan parallèle peut être engendré par les vecteurs d'une base normale composée de p vecteurs deux à deux orthogonaux; pour un p-plan donné, les vecteurs nuls d'une base normale engendrent la partie nulle du p-plan, qui ne dépend que de ce p-plan; si cette partie nulle est à q dimensions, le p-plan considéré est de nullité q. À tout p-plan parallèle correspond le $(n-p)$-plan parallèle conjugué, totalement orthogonal au précédent et de même partie nulle que lui. Dans son papier précédent [loc. cit.], l'auteur a montré dans le cas $n=4$ que l'existence d'un 1-plan parallèle nul entraîne l'existence de plans parallèles autres que son conjugué. Le but principal du présent papier est de montrer qu'il en est ainsi quel que soit n, pour p et q convenables. Dans une première partie, l'auteur reprend l'étude de Y. C.

Wong [Ann. of Math. (2) **46**, 158–173 (1945); ces Rev. **6**, 188] sur les ennuples quasi-orthogonaux (partiellement nuls) pour l'adapter au but poursuivi. Au moyen de cette étude, une forme générale réduite pour les équations de récurrence d'un p-plan parallèle de nullité q est donnée, au moyen d'un ennuple quasi-orthogonal construit en complétant une base normale du p-plan. Les formules ainsi obtenues sont appliquées à deux cas importants: d'abord, dans l'hypothèse où n est pair, le cas d'un $\frac{1}{2}n$-plan de nullité $(\frac{1}{2}n-1)$; dans ce cas il existe un faisceau de $\frac{1}{2}n$-plans parallèles qui admettent en général la nullité $(\frac{1}{2}n-1)$ (et même partie nulle que le plan donné), à l'exception de deux d'entre eux qui sont complètement nuls. Dans l'hypothèse où n est impair, l'existence d'un $\frac{1}{2}(n-1)$-plan de nullité $\frac{1}{2}(n-3)$ entraîne l'existence de deux $\frac{1}{2}(n-1)$-plans parallèles complètement nuls. Dans les deux hypothèses, il s'ajoute bien entendu aux plans parallèles signalés ceux qui s'en déduisent par conjugaison ou passage à la partie nulle. Les résultats signalés par l'auteur pour $n=4$ se trouvent ainsi largement généralisés. *A. Lichnerowicz* (Paris).

Shanks, E. Baylis. Homothetic correspondences between Riemannian spaces. Duke Math. J. **17**, 299–311 (1950).

Le but de l'auteur est l'étude locale des variétés riemanniennes V_n et V_n^* admettant entre elles une correspondance homothétique, c'est-à-dire une correspondance localement biunivoque telle qu'aux points correspondants, les distances des points de V_n et de V_n^* soient dans le même rapport constant. Une correspondance homothétique est au fond une correspondance qui est à la fois conforme et géodésique. L'auteur forme, selon la méthode classique d'Eisenhart, des conditions nécessaires et suffisantes pour qu'il existe une telle correspondance entre V_n et V_n^*. Il étudie spécialement les variétés V_n qui admettent une représentation homothétique sur elle-même, appartenant à un groupe continu à un paramètre. Des formes canoniques par le ds^2 d'une V_n admettant un groupe continu à un paramètre de transformations homothétiques en elle-même sont indiquées. Lorsqu'il en est ainsi, le groupe complex des transformations homothétiques de V_n en elle-même s'obtient par composition du groupe à un paramètre et du groupe complex des déplacements de l'espace. L'auteur étudie aussi le cas où les trajectoires du groupe à un paramètre de transformations homothétiques de V_n en elle-même soit des géodésiques de V_n. La dernière partie du papier est consacrée à l'étude exhaustive du cas $n=2$. Les résultats assez compliqués ne peuvent être résumés ici.

A. Lichnerowicz (Paris).

Rozenfel'd, B. A. On unitary and stratified spaces. Trudy Sem. Vektor. Tenzor. Analizu **7**, 260–275 (1949). (Russian)

A unitary space K_n of Schouten is a complex metric space with coordinates $X^i=\xi^{2i-1}+i\xi^{2i}$ with a real metric determined by a Hermitian tensor $A_{i\bar{k}}=\bar{A}_{\bar{k}i}$ and affine connection whose components with mixed indices are all zero and $\Gamma^k{}_{ij}=A^{\bar{m}k}\partial A_{j\bar{m}}/\partial X^i$; $\Gamma^{\bar{k}}{}_{\bar{i}\bar{j}}=A^{\bar{k}i}\partial A_{\bar{i}j}/\partial \bar{X}^{\bar{i}}$; $\bar{\Gamma}^k{}_{ij}=\Gamma^{\bar{k}}{}_{\bar{i}\bar{j}}$. A stratifiable space of Rashevski is a pseudo-Riemannian space V_{2n} containing 2 families of n-dimensional spaces X_n satisfying the conditions: (1) Through each point of V_{2n} there passes one and only one X_n of each family; (2) the X_n are isotropic; and (3) each X_n is a space of absolute parallelism. This implies that each X_n is totally geodesic in V_{2n}. The present paper is concerned with the question of when is V_{2n} a K_k and conversely. The distinction is made between simply stratifiable spaces satisfying (1) and (2) and absolutely stratifiable spaces satisfying (1), (2), and (3) and the main results are given by the theorem: A real $2n$-dimensional Riemannian space generated by a K_n without torsion is absolutely stratifiable and conversely. The author also gives a number of illustrations of absolutely stratifiable spaces with Riemannian and pseudo-Riemannian metrics.

M. S. Knebelman (Pullman, Wash.).

Sasaki, Shigeo. On the real representation of spaces with Hermitian connexion. Sci. Rep. Tôhoku Univ., Ser. 1. **33**, 53–61 (1949).

L'auteur considère une variété C_n à structure analytique complexe (dimension complexe n, dimension topologique $2n$), sur laquelle se trouve définie une connexion hermitique. La variété C_n peut être interprétée comme variété différentiable à $2n$ dimensions réelles. L'auteur se préoccupe tout d'abord de savoir comment une variété à $2n$ dimensions réelles $C^*{}_{2n}$, munie d'une connexion affine peut être caractérisée comme la représentation réelle d'un C_n à connexion hermitique. L'étude est purement locale et $C^*{}_{2n}$ est lui-même muni d'une structure analytique (autant que le rapporteur a pu le comprendre). Si Γ^A_{BC}, $A=1, \cdots, 2n$, est la connexion affine définie sur $C^*{}_{2n}$ et en général non symétrique nous appellerons symétrique de cette connexion, la connexion ayant pour coefficients $\bar{\Gamma}^A_{BC}=\Gamma^A_{CB}$. Cela posé, le résultat principal est le suivant: Pour qu'un $C^*{}_{2n}$ soit la représentation réelle d'un espace C_n à connexion hermitique, il faut et il suffit que les deux groupes d'holonomie homogènes associés à la connexion donnée et à sa symétrique laissent invariante une collinéation fondamentale I définie par $I_\beta{}^\alpha=I^{n+\alpha}_{n+\beta}=0$; $I^\alpha{}_{n+\beta}=-I_\beta{}^{n+\alpha}=\delta_\beta{}^\alpha$, $\alpha, \beta=1, \cdots, n$. Dans une seconde partie, l'auteur suppose que sur C_n se trouve définie une métrique hermitique ds^2. Une connexion hermitique s'en déduit dont l'auteur étudie la représentation réelle. La connexion affine réelle correspondante laisse invariante la métrique riemannienne à $2n$ variables réelles définie par ds^2, mais n'est pas en général symétrique et, par suite, ne coïncide pas en général avec la connexion riemannienne; cette connexion affine laisse aussi invariante la forme quadratique extérieure définie en antisymétrisant la forme ds^2 (et naturellement la collinéation fondamentale I). Une étude relative aux tenseurs de courbure termine le papier. *A. Lichnerowicz* (Paris).

Yano, Kentaro. Note on the conformal theory of curves. Tensor N.S. **1**, 6–13 (1950).

This paper examines the conformal theory of curves as originally developed by the reviewer [Proc. Nat. Acad. Sci. U. S. A. **26**, 437–439 (1940); Trans. Amer. Math. Soc. **51**, 435–501 (1942); these Rev. **2**, 21; **3**, 307] from the standpoint of the independent work of the author and Mutô. Starting with the author's conformal parameter, it is shown how one may develop the conformal theory of curves up to the derivation of the conformal Frenet equations. Some priority claims are made in the paper which appear to have overlooked the first cited paper and earlier abstracts of the reviewer. *A. Fialkow* (Brooklyn, N. Y.).

Süss, Wilhelm. Ein affines Analogon zur Bestimmung einer Fläche aus einer Grundform und einer Krümmungsfunktion durch W. Scherrer. Math. Z. **52**, 698–702 (1950).

L'autore dimostra che una superficie è univocamente determinata da una sua striscia, non asintotica, quando si

assegnino inoltre, quali funzioni dei parametri; la forma quadratica fondamentale della geometria centro-affine, e la "distanza affine" dell'origine delle coordinate dalla superficie; ovvero: quando si diano inoltre la analoga forma quadratica fondamentale delle affinità equivalenti, e la curvatura media affine. L'autore dimostra l'enunciato quale caso particolare di un teorema più generale della geometria centro-affine, in cui si sfrutta l'idea di una "geometria differenziale relativa ad una superficie Σ di appoggio" [tale concetto è stato introdotto da E. Müller [Monatsh. Math. Phys. **31**, 3–19 (1921)]; vedi anche E. Salkowski [Affine Differentialgeometrie, de Gruyter, Berlin-Leipzig, 1934, 174 e segg.] e la teoria delle coppie di superficie S, S', polari reciproche; l'autore dimostra precisamente che due striscie non asintotiche A_S, A_Σ individuano due superficie per esse, S e Σ, rispettivamente, quando si assegnino, quali funzioni dei parametri, la forma quadratica fondamentale della geometria centro-affine per la superficie S (e per la sua polare reciproca, che è anch'essa univocamente determinata), e due funzioni P, Q, delle quali l'una è, in sostanza, la distanza affine dell'origine dalla superficie S, e l'altra è la somma dei raggi principali di curvatura di Σ "relativamente alla superficie S, quale superficie di appoggio." Identificando le due superficie S e Σ (e tenendo conto che allora Q si riduce alla costante $+2$), si ottiene la prima parte del teorema; la seconda parte si ha, sfruttando le relazioni fra la geometria centro-affine e quella delle affinità equivalenti. Il risultato dell'autore costituisce l'analogo affine di un teorema dimostrato per la geometria differenziale metrica da W. Scherrer [Comment. Math. Helv. **22**, 366–381 (1947); questi Rev. **9**, 464]. *V. Dalla Volta.*

Cossu, Aldo. Sulla curvatura delle varietà a tre dimensioni dotate di una connessione affine. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 551–556 (1950).

Let $R^\nu{}_{\omega\mu\lambda}$ be the curvature tensor of an A_3, v^ν an arbitrary vector, $dx^{[\omega}\delta x^{\mu]}$ an arbitrary bivector. The author considers the projectivities

(1a) $$\rho^* v^\nu = R^\nu{}_{\omega\mu\lambda} dx^{[\omega}\delta x^{\mu]} v^\lambda,$$

where $\rho^* v^\nu$ is obviously the increment suffered by v^ν if transported by a parallelism along an infinitesimal circuit in $dx^{[\omega}\delta x^{\mu]}$. The corresponding equation

(1b) $$(R^\nu{}_{\omega\mu\lambda} dx^{[\omega}\delta x^{\mu]} - \rho\delta_\lambda{}^\nu)v^\lambda = 0$$

for self-preserved directions has in general three solutions. If, in particular, not only the direction of v^ν but the vector v^ν itself is preserved by (1) (e.g., if $\rho=0$), then this direction is called a principal axis (while the other self-preserved directions are termed secondary axes). We have one or more principal axes if the characteristic equation of (1) has a single or multiple root $\rho=0$ and this requirement may be easily expressed in terms of $R^\nu{}_{\omega\mu\lambda}$ and its geometrical consequences. *V. Hlavatý* (Bloomington, Ind.).

Kanitani, Jôyô. Sur l'espace à connexion projective majorante. II. Jap. J. Math. **19**, no. 4, 395–403 (1948).

[For part I see the same vol., no. 3, 343–361 (1947); these Rev. **11**, 54.] In a previous [unavailable] paper the author proved in two ways that there is a projective space S_N $(N=\frac{1}{2}n(n+1)+n-1)$ containing a given space R_n with a symmetric projective connection. In the first proof the method by which Cartan showed the possibility of imbedding a Riemannian space in a Euclidean space was applied. The notion of the dominant connection was used in the second proof, of which some part is incorrect. The purpose of this paper is first to amend the second proof and then to study the variety V_n to which the space R_n is carried by the imbedding. The results can be stated as follows. A space R_n with a symmetric projective connection can be imbedded in a projective space S_n in such a way that this space R_n becomes a variety V_n generated by ∞^{n-1} straight lines and that at any point of V_n the osculating planes of all curves on V_n through this point determine the space S_N. Any ∞^{n-1} geodesics of R_n can be taken as the curves corresponding to the generators of V_n. Once these geodesics are designated, the variety V_n depends on $\frac{1}{2}N(i-1)(i+2)$ arbitrary functions of $n-i+1$ variables, where $i=1, \cdots, n$. *C. C. Hsiung* (Cambridge, Mass.).

Michal, A. D., and Mewborn, A. B. General projective differential geometry of paths. Compositio Math. **8**, 157–168 (1950).

In previous papers, Michal has outlined a general projective geometry of paths for the case in which the coordinate Banach space B was assumed to have an inner product, and a contraction operation was postulated for the ring of linear transformations on B to itself [Proc. Nat. Acad. Sci. U. S. A. **23**, 546–548 (1937); Bull. Amer. Math. Soc. **45**, 529–563 (1939); these Rev. **1**, 29]. The present approach is somewhat different. It is based upon two assumed elements of structure, a linear connection and a gauge form together with their projective laws of transformation. Let B_1 be the Banach space $B\times R_1$, where R_1 is the real line. A new geometric object, called a projective connection, whose components $\Pi(X, Y, Z)$ are functions with arguments and values in B_1, is defined in terms of the postulated linear connection and gauge form. With the introduction of the projective normal parameter π, the paths are represented by the simple differential equation $d^2X/d\pi^2+\Pi(X, dX/d\pi, dX/d\pi)=0$. Finally, the curvature form is introduced, whose components depend upon Π and its first differential. The locally flat case, in which the curvature form vanishes identically, has been developed elsewhere by the same authors [Acta Math. **72**, 259–281 (1940); these Rev. **2**, 166]. *D. H. Hyers* (Los Angeles, Calif.).

NUMERICAL AND GRAPHICAL METHODS

Wood, Harley. Kepler's problem—the parabolic case. J. Proc. Roy. Soc. New South Wales **83**, 181–194 (1950).

The author gives a seven place table of the solution of $12\mu+\mu^3=D$ for $0\leqq D\leqq 100$ at interval 0.1 of D and a six place table for $100<D\leqq 1000$ at interval 1.0. A scheme is outlined for determining the solution if $D>1000$. *R. G. Langebartel* (Saltsjöbaden).

***Tables for Conversion of X-ray Diffraction Angles to Interplanar Spacing.** National Bureau of Standards, Applied Mathematics Series, No. 10. United States Government Printing Office, Washington, D. C., 1950. v+159 pp. $1.75.

Tables of $\lambda/2\sin\theta$ for $\lambda=0.70926$, 1.54050, 1.65783, 1.78890, 1.93597, 2.28962, and $\theta=0(.01)90°$; 5S. Also a

rearrangement of the tables for $\lambda=1.54050$, 1.93597 for $2\theta=0(.02)180°$; 5S. The values of λ were chosen for their physical significance.

Miller, J. C. P. Checking by differences. I. Math. Tables and Other Aids to Computation **4**, 3–11 (1950).

L'auteur donne des détails sur la manière de vérifier une table par différences. Il rappelle l'influence sur les différences des divers ordres de l'arrondissement d'une erreur isolée. Il indique comment localiser et corriger une erreur isolée. Il discute l'ordre de grandeur des erreurs detectables, la borne que doivent dépasser les différences pour qu'il y ait certainement une erreur. Il y a interet à remplacer cette borne par une autre plus stricte, qui conduit à réexaminer environ une différence sur cent. Dans un prochain article l'auteur traitera des erreurs non isolées et des erreurs commises en formant les différences. *J. Kuntzmann.*

Taylor, Norman H. Marginal checking as an aid to computer reliability. Proc. I.R.E. **38**, 1418–1421 (1950).

Collatz, L. Über die Konvergenzkriterien bei Iterationsverfahren für lineare Gleichungssysteme. Math. Z. **53**, 149–161 (1950).

The author establishes as a basic tool, the following theorem: Consider the vector equation $Ax=r$ and express A as $A=B+C$, where B has an inverse. Let $F=-B^{-1}C$. Then the root x can be obtained as $B^{-1}r+FB^{-1}r+F^2B^{-1}r+\cdots$ provided all roots K of $\det(KB+C)=0$ are such that $|K|<1$. It is customary to take B either as the diagonal part of A or, in the Gauss-Seidel process, the diagonal and subdiagonal part. Using this theorem the author establishes the convergence of the Gauss-Seidel method in the case in which A is positive definite. He also investigates convergence criterions involving a comparison between the diagonal elements and the sum of the absolute values of the remaining elements in a row or column, including the "weak" case of equality for certain columns. He points out applications of these latter to the solution of ordinary differential equations with boundary conditions and to problems on elliptic partial differential equations. *F. J. Murray.*

Woodbury, Max A. Inverting modified matrices. Statistical Research Group, Memo. Rep. no. **42**, Princeton University, Princeton, N. J., 1950. 4 pp.

Le problème indiqué est résolu par les formules:

$$(A+USV)^{-1}=A^{-1}-A^{-1}US(S+SVA^{-1}US)^{-1}SVA^{-1},$$
$$(A-US^{-1}V)^{-1}=A^{-1}+A^{-1}U(S-VA^{-1}U)^{-1}VA^{-1},$$

où A est la matrice carrée d'ordre $n\times n$, $A+USV$ la matrice modifiée (U, V matrices d'ordre $n\times m$ et $m\times n$, S matrice carrée d'ordre $m\times m$, m petit devant n). On utilise la première formule ou la seconde suivant que l'on ne connait pas ou que l'on connait l'inverse de S. Les opérations necessaires sont des produits de matrices et l'inversion d'une matrice carrée d'ordre $m\times m$. Ces formules ne sont pas au fond différentes de celles utilisées dans l'inversion d'une matrice formée de sous matrices. *J. Kuntzmann.*

Taussky, Olga. Notes on numerical analysis. II. Note on the condition of matrices. Math. Tables and Other Aids to Computation **4**, 111–112 (1950).

In connection with inverting a matrix A, one can use as a figure of merit for the accuracy of the inverse matrix one of several criteria. The author in this note chooses two: the P-condition number which is the absolute value of the ratio of the largest to the smallest characteristic value of A; and the N-condition number which is the product of the norm of A by the norm of A^{-1} divided by the order of the matrix. In terms of these figures of merit the author shows that the P- and N-condition numbers of A are not greater than those for AA', where A' is the transpose of A. *H. H. Goldstine* (Princeton, N. J.).

Forsythe, George E., and Leibler, Richard A. Matrix inversion by a Monte Carlo method. Math. Tables and Other Aids to Computation **4**, 127–129 (1950).

The authors in this paper give an exposition of a method for inverting matrices devised by von Neumann and Ulam based on stochastic techniques. Under suitable hypotheses the authors are able to exhibit the desired game and establish its relevant properties. In general, the method is for each i, j to play a solitaire game whose expected payment is the (i, j)-th element of the inverse matrix. This expectation value is approximated by playing successively and calculating the average payment. *H. H. Goldstine.*

Čarnyĭ, I. A. On the movement of ground water into gas deposits of dome type. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1950**, 1326–1344 (1950). (Russian)

This paper is concerned with the development and numerical solution (Picard's method) of a differential equation of the type $dv/dt=\Phi(t, v)$, arising due to forcing of a gas from an underground deposit. Novelty is claimed in permitting influence of gravity and elasticity of the water to enter the problem. Considerable space is devoted to numerical calculations. *R. E. Gaskell* (Ames, Iowa).

Thomas, L. H. Stability of solution of partial differential equations. Symposium on theoretical compressible flow, 28 June 1949. Naval Ordnance Laboratory, White Oak, Md., Rep. **NOLR-1132**, pp. 83–94 (1950).

Analysis of the stability (or error growth) of finite difference equivalents of ordinary and partial differential equations has become a problem of major practical importance in recent years due to the application of finite difference methods for the numerical integration of systems of differential equations on high-speed calculators. It was first shown by Courant, Friedrichs, and Lewy [Math. Ann. **100**, 32–74 (1928)] that the use of certain finite difference representations do not yield the true solutions of the corresponding partial differential equations, no matter how small the mesh size is chosen. More recently J. von Neumann has contributed extensively to the analysis of error growth for finite difference equivalents of differential equations. A simple illustration of the difficulties which may be encountered by the use of finite differences is illustrated by the finite difference equivalent

$$\frac{T(x, t+\Delta t)-T(x, t)}{\Delta t}=\kappa\frac{T(x+\Delta x, t)-2T(x, t)+T(x-\Delta x, t)}{(\Delta x)^2}$$

of the heat conduction equation $\partial T/\partial t=\kappa\partial^2T/\partial x^2$. In this particular case, no reasonable solution can be obtained if Δt is chosen so that $\Delta t>\frac{1}{2}\kappa(\Delta x)^2$. A solution based on such a choice of Δt is unstable and will grow exponentially beyond bounds.

The author presents a survey of the present status in the art of stability analysis of finite difference equivalents of differential equations. He first considers in some detail the first order ordinary differential equation $dy/dx=f(x, y)$. He makes an analysis of the growth of error, both by the use

of a finite difference equivalent based on Simpson's rule of integration and one based on the Gregory integration formula (correct to third differences). The stability characteristics of several common types of partial differential equations are considered next. The following equations are discussed: $F(\partial z/\partial x, \partial z/\partial y, x, y, z)=0$; $\partial^2v/\partial t^2=c^2\partial^2v/\partial x^2$; $\partial v/\partial t=a\partial^2v/\partial x^2$; $\partial^2v/\partial x^2+\partial^2v/\partial y^2=\rho(x, y)$. The analysis used by the author relates the growth of error to the absolute values of the characteristic roots of the finite difference equations under consideration. The author defines two types of instability: short-range instability, in which the absolute value of the characteristic root is greater than unity; and long-range instability, in which the absolute value of the characteristic root is exactly equal to unity. On the basis of the analysis carried out for the above types of differential equations the author makes several conjectures concerning the occurrence of long-range or short-range instability for more general systems of equations.

H. Polachek (White Oak, Md.).

Faddeeva, V. N. The method of lines applied to some boundary problems. Trudy Mat. Inst. Steklov. **28**, 73–103 (1949). (Russian)

Poisson's equation $\nabla^2u(x, y)=f(x, y)$ is to be satisfied in some region R, with specified boundary conditions. If $y_k=y_0+kh$, $u_k(x)=u(x, y_k)$, $f_k(x)=f(x, y_k)$, then the set of equations

$$(*)\quad 10u_k''+u_{k-1}''+u_{k-1}''+12h^{-2}[u_{k+1}-2u_k+u_{k-1}]-12F_k=0$$

with $F_k=[10f_k(x)+f_{k+1}(x)+f_{k-1}(x)]/12$ is a replacement for Poisson's equation proposed by Slobodiansky [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] N.S. **3**, no. 1, 75–82 (1939)]. The author writes (*) in matrix form (**) $AU''+Mh^{-2}U-F=0$, where U and F are column matrices with elements u_k and F_k, respectively. The coefficient matrices are related by $A=I+M/12$. A symmetric orthogonal matrix B is then devised such that $B^{-1}MB=\Lambda$ is a diagonal matrix. Then $B\Lambda B^{-1}=M$, $B(I+\Lambda/12)B^{-1}=A$, and since $B=B^{-1}$, (**) can be written $(I+\Lambda/12)V''+h^{-2}\Lambda V-G=0$, where $V=BU$ and $G=BF$ are also column matrices. From this last matrix equation one gets equations of the type $(1+\lambda_k/12)v_k''(x)+\lambda_kh^{-2}v_k(x)-g_k(x)=0$ $(k=1, \cdots, n)$, which can be solved individually. Several examples are given to illustrate the method. These include the torsion problem for rectangle, isosceles trapezoid, ellipse, and semicircle. Some discussion is given for other types of equations, including the wave equation and the heat equation in one dimension.

R. E. Gaskell (Ames, Iowa).

Wang, Chi-Teh, and Brodsky, R. F. Approximate solution of compressible fluid-flow problems by Galerkin's method. J. Aeronaut. Sci. **17**, 660–666 (1950).

Hodgson, M. L., Clews, C. J. B., and Cochran, W. A punched-card modification of the Beevers-Lipson method of Fourier synthesis. Acta Cryst. **2**, 113–116 (1949).

"A description is given of a punched-card system for Fourier synthesis which may be regarded as a mechanized version of the original Beevers-Lipson method, and which has been used successfully for the evaluation of two- and three-dimensional Fourier series."

Authors' summary.

Cox, E. G., Gross, L., and Jeffrey, G. A. A Hollerith technique for computing three-dimensional differential Fourier syntheses in X-ray crystal-structure analysis. Acta Cryst. **2**, 351–355 (1949).

Booth, A. D. The physical realization of an electronic digital computor. Electronic Engrg. **22**, 492–498 (1950).

Singleton, Henry E. A digital electronic correlator. Proc. I.R.E. **38**, 1422–1428 (1950).

Archer, A. A Venn diagram analogue computer. Nature **166**, 829 (1950).

Scott, R. E. An analog device for solving the approximation problem of network synthesis. Research Laboratory of Electronics, Massachusetts Institute of Technology, Tech. Rep. No. **137**, i+47 pp. (1950).

***Allcock, H. J., Jones, J. Reginald, and Michel, J. G. L. The Nomogram. The Theory and Practical Construction of Computation Charts.** 4th ed. Pitman Publishing Corp., New York, Toronto, London, 1950. x+238 pp. $3.75.

The third edition was published in 1941. The principal change in this edition, prepared by the third author of the title, is the addition of a chapter entitled, "Connection between alignment and intersection nomograms."

Graf, U., und Henning, H. J. Drei Nomogramme zur Bestimmung von Mittelwert-Toleranzen. Mitteilungsblatt Math. Statist. **2**, 90–92 (1 plate) (1950).

Bachmann, W. K. Calcul symbolique des coefficients de poids et de corrélation des inconnues dans le cas d'observations médiates ou conditionnelles. Bull. Tech. Suisse Romande **76**, 74–77 (1950).

This paper is a contribution to the so-called "theory of adjustment." The author claims no novelty for the "adjustment" equations given in his paper, but does state that his derivation of them is novel in that it employs a symbolic calculus due to J. M. Tienstra. The procedures are formal. The author's notation is complicated and not well-explained. A much clearer description of the problem can be found in Arley and Buch, Introduction to the Theory of Probability and Statistics, chapter 12 [Wiley, New York; Chapman & Hall, London, 1950; these Rev. **11**, 187].

B. Epstein (Detroit, Mich.).

Stelson, H. E. The evaluation of varying annuities at varying rates of interest. Giorn. Mat. Finanz. (3) **8**, 14–19 (1950).

Es wird der Endwert einer einmaligen Einzahlung angegeben, wenn diese zu einem Zinsfuss i_1, die dadurch anfallenden jährlichen Zinsen (d.h. i_1) zu einem Zinsfuss i_2, die Zinsen an diesen wiederum zu einem Zinsfuss i_3, usw. angelegt werden. Eine analoge Formel wird für den Endwert gewöhnlicher Annuitäten entwickelt, schliesslich auch für den Fall, dass diese selbst veränderlich sind.

P. Thullen (Panamá).

MECHANICS

Čerkudinov, S. A. On a family of double-crank four-hinge linkages. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **2**, 150–155 (1947). (Russian)

The extreme angular velocities (a.v.) ω_1, ω_2 of the driven member O_BB of a four-hinge linkage (for a constant a.v. ω of the driving member O_AA) occur when the instant-center line TP is normal to the connecting rod $AB=b$ ($T=AB\times O_AO_B$; $P=O_AA\times O_BB$). This corrects some older erroneous statements [Kraus, Maschinenbau **18**, 37–41, 93–94 (1939)]. A class of mechanisms is defined by the condition that the extremes occur when $AB\perp O_AO_B$, while $O_AA=O_BB$. If $O_AA=a$, $O_AO_B=l$, $a>l$, $AB=b$, then $b^2=l(l+2a)$, $\omega_1\omega_2=\omega^2$. Both cranks turn through the same angle α between the extremes. A tabulation of the corresponding values of a/l, b/l, ω_1/ω, ω_1/ω_2, and ABO_B is given.

A. W. Wundheiler (Chicago, Ill.).

Čerkudinov, S. A. On the extremal velocities of slider-crank mechanisms. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **2**, 156–163 (1947). (Russian)

Continuing the paper reviewed above, the author studies the extrema i of the transmission ratio between the members of the turning-block linkage with an offset O_oACO [O_o, O are fixed hinges; A a hinge sliding along AC; $AC\perp CO$; CO the offset]. Let $AT\perp AC$, $T=AT\times OO_o$, $P=O_oA\times OC$, $Q=O_oO\times AC$ (the cross indicates intersection). Then, if $PT\parallel AC$, T is an extreme position of the instant center of AC relative to O_oA because its velocity is zero. If i is the corresponding transmission ratio, then $O_oQ=OO_oi^2/(i-1)^2$. Constructions for a given i (max or min) are presented, and the relation $2i_{\max}i_{\min}=i_{\max}+i_{\min}$ derived. The value i of either $i_{\max}$ or $i_{\min}$ defines the linkage dimensions. For $i<\frac{1}{2}$ a quick-return mechanism (Whitworth), $O_oA<OO_o$, is obtained. For $i>\frac{1}{2}$ both members revolve. The case $i=\frac{1}{2}$ is singular since $i_{\max}=i_{\min}$ and the transmission ratio is constant. For $i<\frac{1}{2}$ the mechanism gets deadlocked if AC is the driving member. Other differences are noted in the transmission ratio, corresponding with an interchange of the driving and driven members.

A. W. Wundheiler (Chicago, Ill.).

Čerkudinov, S. A. On a method of approximation in the synthesis of mechanisms. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **5**, no. 20, 34–77 (1948). (Russian)

The paper expands a previous version [Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1948**, 1517–1530; these Rev. **10**, 409]. The present version adds (1) utilization of Descartes' rule, (2) more exemplification of special cases ($m=n=2, 3, 4$; see the review of the paper cited above), (3) more detail in the generation of a straight line as a slider-crank curve, and (4) the generation of approximate ellipses and hyperbolas by a slider-crank and a "lambdoid" four-hinge linkage (OO_1AB; OO_1 fixed, D on AB; $OB=BA=BD$; D the tracing point). There are some polemics with Bloh's modified Čebyšev method [Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestiya Akad. Nauk SSSR] **1946**, 683–696; these Rev. **8**, 100], the main objection being that no "smallness" of $q(x, y)$ guarantees that (x, y) is close to the curve $q(x, y)=0$. In the author's method a maximum of true intersections with the generated curve is required.

A. W. Wundheiler (Chicago, Ill.).

Dobrovol'skiĭ, V. V. The synthesis of spherical mechanisms. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **1**, 5–20 (1947). (Russian)

The stereographic representation is applied to the theory of spherical mechanisms. After a review of the basic constructions involving "straight lines" and "circles" in the stereographic plane, the method is applied to establish the (already known) basic analogues of the (Burmester) theory of n plane positions ($n=2, 3, 4, 5$). A list of problems treated follows: the quadratic correspondence (defined by the pole triangle) between a point M and the center O_M of the circle C_M of its three positions; the locus of the M's for a given radius of C_M; the design of a spherical four-hinger when three crank-and-lever positions are given; the locus of O_M for variable M; points whose four positions are collinear or cocircular; four-hingers taking a sphere through four given positions, or yielding four given crank-and-lever positions; four-hingers taking a sphere through five given positions; geometry of trajectories in general spherical motion: the locus of points of given trajectory curvature; the locus of third and fourth order points, of Ball's points, and of Burmester's points.

A. W. Wundheiler (Chicago, Ill.).

Dobrovol'skiĭ, V. V. Spherical representation of three-dimensional four-bar linkages. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **2**, 111–126 (1947). (Russian)

Dobrovol'skiĭ, V. V. The method of spherical representation in the theory of spatial mechanisms. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **3**, no. 11, 5–37 (1947). (Russian)

In the second paper a systematic enumeration is given of one-loop spatial n-bar linkage mechanisms with n lower kinematic pairs ($n=3$ to 7); the pairs have not more than three degrees of freedom. For each of the mechanisms the following questions are treated: (1) the existence of passive constraints; (2) the determination of the positions; (3) the determination of the velocities; (4) the kinetostatic (dynamic) equations. The method consists of solving vector equations expressing the vanishing of the sum of the (relative) instantaneous velocity screws. The directions of the vectors are known in part or completely. The equations are solved by means of a representation of the vectors by the tips of parallel unit vectors, centered at the same point, and tagged with the length of the vector. The first ten pages of the paper are devoted to the basic constructions in terms of this representation.

The first paper gives in more detail certain developments of the second one: (1) the (known) relations between the two crank angles of a spherical four-hinger; (2) the theory of the (degenerate) four-bar linkage with four sliding pairs; (3) the mechanism with one turning and three cylindrical pairs; (4) Bennett's mechanism; (5) conditions for four-linkers with two, three, and four helical pairs; (5) the mechanism with one spherical, one cylindrical, and one turning pair.

A. W. Wundheiler (Chicago, Ill.).

Koževnikov, S. N. On the kinematics and design of spatial crank-lever mechanisms. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **4**, no. 14, 32–63 (1948). (Russian)

A detailed study is made of a spatial four-bar mechanism, where A and D are two fixed turning pairs, B and C are

two higher pairs (of stated design) of mobility two, the connecting-rod BC being prevented from rotating about its own axis. The angle between the planes H and V, in which B and C move, is denoted by α. The construction of the position of BC is given in terms of the projections of the mechanisms on H and V. The condition for the existence of a revolving crank is given in terms of $|BC|$, and the distances between the extreme positions of B and C. The velocity of C is determined from that of B by constructions performed on projections onto H and V. The same is done for the acceleration of C. Extreme positions of CD occur when BC is in the plane through AB and the axis of rotation of AB. The corresponding construction in the H and V planes are given, as well as the one for the angle of oscillation of CD. The analytical determination of the latter quantity is also given. The paper concludes with an analytical design of the mechanism for $\alpha=90°$, given center distance, given swing angle of DC, and the ratio of its travel and return times. *A. W. Wundheiler* (Chicago, Ill.).

Tavhelidze, D. S. Concerning the existence of a crank or two cranks in spatial mechanisms. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **3**, no. 9, 5–17 (1947). (Russian)

The same mechanism as in the paper reviewed above is considered, but an analytical treatment is favored. The trigonometry of the linkage is derived, and the conditions for a revolving crank obtained explicitly in terms of the dimensions of the mechanism (which the preceding paper does not do). Special cases are considered. In general, the pairs B and C (A and D are the fixed turning pairs) must have mobilities equal to two and three. For special degenerate cases (as spherical or Bennett's mechanisms) these mobilities can be reduced to unity. The author derives the corresponding conditions. *A. W. Wundheiler.*

Levitskiĭ, N. I. Symmetric trajectories of many-bar linkages. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **4**, no. 13, 5–41 (1948). (Russian)

Levitskiĭ, N. I. Asymmetric trajectories of many-bar linkages. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **4**, no. 15, 5–19 (1948). (Russian)

A recurrence procedure for deriving trajectory equations for certain points of an extensive class (L) of linkages is presented, the first paper dealing with an unessentially restricted subclass of (L). The linkages are obtained by successive adjunctions of "dyads" consisting of two bars, OB and BM, B being a sliding or a turning pair. On adjunction, M is hinged to a linkage point of known trajectory, while O is either hinged to a fixed point, or slid on a fixed straight line. If x_1, y_1 are the coordinates of a point M_1 solid with BM, and x, y the coordinates of M, relations of the form $x=f(x_1, y_1)$, $y=g(x_1, y_1)$ exist, and whenever $\phi(x, y)=0$ is known to hold for M, an equation $\phi_1(x_1, y_1)=0$ can be derived. This is the recurrence procedure.

The second paper derives f and g for the cases (1) when O and B are both hinges, (2) when $O=B$ slides and B is a hinge, and (3) when BM slides through a fixed point O. In the cases (1) and (3), if the trajectory of M is algebraic of degree n, that of M_1 will be, in general, of degree $4n$. But if the first of the points M moves on a circle, the degree will increase three times only, and the degree will be $2\cdot 3^{n-1}$ for an n-bar linkage. If the initial M moves on a straight line, the degree increases by $2\cdot 3^{m-1}$, if m is the number of bars being increased. In case (2), the degree of the trajectory is doubled if the initial M moves either on a circle or on a straight line. Most of these theorems are derived in the first paper under some unnecessary symmetry restrictions relaxed in the second paper. Conditions for symmetric trajectories are scattered over the two papers. *A. W. Wundheiler* (Chicago, Ill.).

Carter, W. J. Acceleration of the instant center. J. Appl. Mech. **17**, 142–144 (1950).

The acceleration of the members of a mechanism with a "floating link" (that means a link of which no end describes a circular path) are not obtainable by the method of relative motion. The author gives a solution of this problem making use of the acceleration of the instant center of velocity. The method developed may also be used to obtain the accelerations for linkages which normally require the use of Coriolis' law. [In the first part of the paper the author shows that the acceleration of the center is $\mu\omega$, where μ is the velocity of the center and ω the angular velocity of the system. This result is well known and may be found in many textbooks on kinematics [see, e.g., Polster, Kinematik, Göschen, Leipzig-Berlin, 1912, p. 37; Beyer, Technische Kinematik . . . , Barth, Leipzig, 1931, pp. 246–249].] *O. Bottema* (Delft).

Sobrero, Luigi. Sui "meccanismi calcolatori" di Svoboda. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 480–483 (1950).

Svoboda's methods of designing linkage computing mechanisms [Computing Mechanisms and Linkages, McGraw-Hill, New York-London, 1948; these Rev. **9**, 381] are largely intuitive and empirical. The present paper indicates an analytical approach by showing how a pair of rolling pivoted cams, which impose a given functional relationship between the rotations of two shafts, are replaceable by a connecting-rod which joins two conjugate points, one on each cam. Two points are said to be conjugate if each is the center of curvature of the relative path of the other. The points may be taken on the osculating circles of the two cams so that their separation is constant during an infinitesimal rotation except for infinitesimals of the fifth or higher order. *M. Goldberg* (Washington, D. C.).

Sobrero, Luigi. Di una elementare proprietà cinematica analoga al principio di Fermat. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 360–364 (1950).

The plane motion of one rigid body B_1 rolling on another (fixed) rigid body B_2 is considered. A point P_1 fixed with respect to B_1 and a point P_2 fixed to B_2 are said to be conjugate of order n at a given position, if an infinitesimal motion of B_1 changes the distance P_1P_2 by an infinitesimal of order n. This variational property of the path from P_1 to P_2 is considered to be analogous to Fermat's principle in geometrical optics. Properties and enumeration of such conjugate pairs are discussed. *D. C. Lewis.*

Liebetegger, A., Northover, F. H., and Thwaites, B. The problem of the swing. Math. Gaz. **34**, 84–93 (1950).

Rubbert, Friedrich Karl. Zur Theorie des sphärischen Pendels. Z. Physik **128**, 56–71 (1950).

This essentially expository article contains rather complete information on the solution of the differential equations for a spherical pendulum. The solutions (both approximate and exact in terms of elliptic functions) are exhibited in such a

way as to indicate explicitly the dependence on initial values as well as upon the time. *D. C. Lewis* (Baltimore, Md.).

Haag, Jules. Pendule conique isochrone. C. R. Acad. Sci. Paris **231**, 933–935 (1950).

The author describes a modified and improved form of the Huygens conical pendulum, and shows theoretically that the device can be constructed so as to satisfy modern requirements as to stability and accuracy. A more adequate exposition of the conception will be published later. *L. A. MacColl* (New York, N. Y.).

Morris, J. The whirling of a spinning top. Aeronaut. Quart. **2**, 9–14 (1950).

An elementary and approximate treatment for the small vibrations of the axis of a top. A study of stability is included. *D. C. Lewis* (Baltimore, Md.).

Grioli, Giuseppe. Una proprietà caratteristica delle precessioni regolari del solido pesante asimmetrico. Rend. Sem. Mat. Univ. Padova **19**, 237–248 (1950).

Making use of a special form for the moment of momentum of an arbitrary solid with one fixed point, the author points out a property characteristic of regular precession in terms of harmonic oscillation of the so-called antipole of the axis of rotation with respect to a certain section of the ellipsoid of inertia. *D. C. Lewis* (Baltimore, Md.).

Manacorda, Tristano. Sul moto di un solido attorno ad un punto fisso. Ricerca Sci. **20**, 487–490 (1950).

Let O denote the fixed point and G the center of gravity of the solid. The author seeks to characterize a solid which, acted upon by a system of forces whose moment is perpendicular to OG, has a moment of momentum also perpendicular to OG. Such conditions coincide with those found by Hess [Math. Ann. **37**, 153–181 (1890)], but they are here given a much simpler and more geometric form. *D. C. Lewis* (Baltimore, Md.).

Wood, Harley. Kepler's problem. J. Proc. Roy. Soc. New South Wales **83**, 150–163 (1950).

Solution for the nearly parabolic case is developed as a power series in $(1-e)/(1+e)$ (e is eccentricity) and a discussion of its region of validity given. Considerations of a numerical nature of the general Kepler equation are also given and an extensive bibliography supplementary to previous standard ones is appended. *R. G. Langebartel.*

Černyĭ, S. D. The motion of material points under the influence of forces imparting to them an acceleration $-\mu_1 r^{-2}-3\mu_2 r^{-4}$. Akad. Nauk SSSR. Byull. Inst. Teoret. Astr. **4**, no. 6(59), 287–308 (1949). (Russian)

Among the possible curves traced by a particle moving with this acceleration are cardioids and slightly distorted lemniscates. One form for the equation of motion leads to the periplegmatic orbits of Gyldén [Traité analytique des orbites absolues des huit planètes principales, vol. 1, Stockholm, 1894]. *R. G. Langebartel* (Saltsjöbaden).

Ura, Taro. On canonical transformations. Jap. J. Astr. Geophysics **21**, no. 3, 55–66 (1947).

Necessary and sufficient conditions are found that a transformation, which need not be a contact transformation, should leave invariant the Hamiltonian form of a given canonical system. Linear canonical transformations are especially considered. *D. C. Lewis* (Baltimore, Md.).

Ura, Taro. On canonical transformations. Proc. Japan Acad. **22**, nos. 1–4, 1–5 (1946).

An abbreviated version of the paper reviewed above.

Miyahara, Shimesu. Extension of the method of Hamilton and Jacobi. Jap. J. Astr. Geophysics **21**, no. 3, 67–85 (1947).

If a partial differential equation of the first order (say, analytic) is semilinear in the sense of Lie, it may not have an easily available complete integral in the ordinary sense (although, contrary to the author's erroneous statement, it might have). It always has, however, a complete integral in the generalized sense of Lie. The author shows how the Hamilton-Jacobi method for integrating the differential equations of a holonomic conservative dynamical problem can be extended so as to work even when the available complete integral of the Hamilton-Jacobi partial differential equation is of the above mentioned generalized type. Secondly, he shows that such a complete integral is always available when the Hamilton-Jacobi equation is semilinear. *D. C. Lewis* (Baltimore, Md.).

Miyahara, Shimesu. On the relation between infinitesimal transformation and integral. Jap. J. Astr. Geophysics **21**, no. 3, 87–118 (1947).

The author considers the general problem of obtaining first integrals for canonical systems which are left invariant by given continuous groups of transformations and vice versa. Since the transformations considered are not necessarily of the so-called contact type, some of the results, even though they belong to a very classical and much studied field, are probably new. The three body problem is used as an illustration. *D. C. Lewis* (Baltimore, Md.).

Landolt, M. Die Tensorkoordinaten des Drehwinkels. Elemente der Math. **5**, 104–113 (1950).

The problem of representing a finite rotation by means of a tensor is treated in various texts on mechanics. In this paper, the author presents a simple and interesting approach to this problem. The essential tools used by the author are the components of the e-pseudo tensor and the components of vectors with respect to a 3-tuple of unit vectors. Since the unit vectors of the 3-tuple are not necessarily orthogonal, contravariant and covariant quantities are used. If ϕ represents the angle (measured perpendicularly to the axis of rotation z^i) through which a vector a^i is rotated, then the author shows that the rotated vector b^i is given by $b_i=\phi_{ik}a^k$, where the rotation tensor ϕ_{ik} is $\phi_{ik}=z_iz_k(1-\cos\phi)+g_{ij}\cos\phi+e_{ijk}z^k\cos\phi$, and g_{ij}, e_{ijk} are the components of the metric tensor and the e-pseudo tensor with respect to the 3-tuple. *N. Coburn.*

Giorgi, Giovanni. Un enunciato generale sulla dinamica dei sistemi. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 175–178 (1950).

The author gives the six equations of motion for a rigid body in a form in which motors are used instead of vectors and considers some applications. *J. Haantjes* (Leiden).

Hydrodynamics, Aerodynamics, Acoustics

✱Birkhoff, Garrett. Hydrodynamics. A Study in Logic, Fact, and Similitude. Princeton University Press, Princeton, N. J., 1950. xiii+186 pp. (2 plates). $3.50.

La science moderne des fluides est devenue si vaste que le titre de ce livre ne renseigne guère le lecteur sur les matières

traitées dans ce volume. Force nous est d'en analyser le contenu chapitre par chapitre.

Au chapitre I, l'auteur rappelle brièvement les axiomes qui servent de fondement aux équations de l'hydrodynamique: il discute la validité de ces prémisses abstraites en passant en revue quelques-unes de leurs conséquences paradoxales dont l'étude constitue la meilleure "introduction à l'art d'appliquer les théories." Ces paradoxes sont de plusieurs types. Les uns tiennent à la nature même des axiomes de base: le mathématicien, pour des raisons banales de commodité, prête aux corps qu'il étudie des propriétés de régularité que les corps réels ne possèdent pas toujours. Mais il peut arriver, et c'est ici que l'examen critique de l'auteur devient vraiment suggestif grâce à la variété et, souvent, à l'originalité des faits invoqués à l'appui de ses thèses, que les conséquences singulières des calculs découlent des propriétés analytiques profondes des équations de la physique mathématique. On sait, par exemple, que les formules valables pour les écoulements des fluides visqueux ne se réduisent pas, quand on fait tendre le coefficient de viscosité vers zéro, à celles qui gouvernent les mouvements des fluides parfaits. D'un autre côté, la structure même des équations de l'hydrodynamique est telle que l'apparente symétrie des causes ne se retrouve pas nécessairement dans les effets [cf. le phénomène bien connu des tourbillons alternés de Kármán]. L'auteur insiste à cette occasion sur l'instabilité physique de certains régimes symétriques théoriquement acceptables pour des valeurs appropriées du nombre de Reynolds. Il est naturel de réserver aux paradoxes, dits de résistance, des développements étendus. Ici l'auteur utilise la notion qui semble féconde, de théorie reversible; ce procédé lui permet d'analyser, dans une certaine mesure, la nature des difficultés auxquelles se heurte la théorie; mais il subsiste là bien des énigmes qui attendent encore leur solution. Signalons que l'auteur discute plusieurs paradoxes découverts récemment [von Neumann, Kopal] et qui, à notre connaissance, n'ont pas encore fait l'objet d'un exposé didactique d'ensemble.

Au chapitre II, l'auteur expose d'abord la théorie classique des sillages et des jets, fondée par Helmholtz et Kirchhof, et développée depuis par Brillouin, Villat, Leray, et Weinstein; il laisse de côté l'examen des méthodes variationnelles, introduites dans ces questions par Lavrentieff. Il discute une fois de plus la validité physique de la théorie en cause et rappelle les services d'ordre pratique qu'elle rend dans la théorie des jets. Dans la deuxième section du chapitre, l'auteur traite des aspects moine traditionnels du problème. Divers auteurs ont été amenés à considérer systématiquement deux paramètres sans dimension ρ'/ρ et Q. Le premier mesure le rapport des densités des fluides en contact le long d'une ligne de discontinuité pour les vitesses; $\rho'/\rho=1$ dans le cas classique et vaut 0.0013 dans le cas d'une veine d'eau s'échappant dans l'air. D'après Betz et Petersohn, la théorie des sillages s'appliquerait aux phénomènes réels chaque fois que ρ'/ρ est petit. Cependant l'auteur signale des cas où cette hypothèse est en défaut. Par exemple, le phénomène de la chute d'une petite sphère à la surface d'une eau tranquille offre deux aspects différents suivant que la poche creusée dans la nappe se referme à la surface ou en profondeur (phénomène de Worthington). La théorie de cette expérience a été faite par Taylor et complétée par Gilbarg et Anderson et l'auteur. On est ainsi conduit à établir l'existence d'une valeur critique de $F^2\rho'/\rho$, F étant le nombre de Froude, prévision que l'expérience confirme. Le second paramètre: $Q=2p/\rho v^2$ (où p est l'excès de la pression à l'infini sur celle qui règne dans le sillage et où v est la vitesse à l'infini) se réduit à zéro dans la théorie classique. En prenant $Q>0$, on arrive à former des modèles hydrodynamiques qui éliminent le paradoxe de Brilloin. L'auteur étudie les schémas s'inspirant de cette idée, construits par Riabouchinski, Gilbarg, Lighthill, et Southwell, en en discutant, comme toujours, la portée physique. Enfin, l'hypothèse des lignes de jet ouvre la voie à l'interprétation d'autres phénomènes connus (effets de paroi, effet des charges creuses, sillages non stationnaires de von Kármán).

Le chapitre III est consacré à la théorie de la similitude qui a pour objet la justification de la méthode des modèles réduits. Une brève introduction fait ressortir l'importance du problème et met en lumière ses difficultés spécifiques. L'auteur décrit ensuite les deux méthodes employées pour attaquer la question. La première est connue sous le nom d'analyse dimensionnelle; l'auteur en fait le rapide historique, en énonce les principaux résultats (entre autre le célèbre théorème de Π, exposé de manière remarquablement suggestive et propre à en faire voir la généralité) et illustre son résumé au moyen de nombreux exemples, empruntés spécialement à la théorie de la résistance. Un examen critique des fondements de la théorie, qui s'inspire des idées de Bridgman, termine ce paragraphe. La deuxième méthode est dite "inspectional analysis." Elle a pour objet propre l'étude des invariants (relativement aux transformations d'un groupe qui sera, pratiquement, toujours celui des similitudes) des équations qui gouvernent un phénomène déterminé. C'est ici que trouve sa place l'analyse des travaux de Afanassjewa-Ehrenfest. Ces généralités sont abondamment commentés par des modèles de similitude utilisés en mécanique des fluides. Pour finir, l'auteur souligne que la théorie des modèles réduits offre un champ spécialement fécond à la collaboration nécessaire du mathématicien et de l'ingénieur.

Le chapitre IV expose les applications de la théorie des groupes au problème de l'intégration des équations de l'hydrodynamique que l'auteur formule comme il suit: étant donné un système différentiel Σ, invariant par les transformations d'un groupe G, trouver les solutions de Σ invariantes par G. On est ainsi ramené à un problème particulier, souvent plus facile et dont la solution entraine, quelquefois, celle du problème général. Le procédé est illustré sur l'exemple de l'équation de la chaleur; mais il a été appliqué aussi avec succès à la théorie des ondes de Prandtl-Meyer, à l'étude des écoulements coniques de Taylor-MacColl. D'une manière générale, les solutions d'un problème de physique mathématique, invariantes par les éléments de G, revêtent une forme spécialement simple (réduction du nombre de variables indépendantes) si on utilise les variables canoniques, dites associées à G. Cette propriété donne à l'auteur d'importants énoncés concernant la forme générale des solutions des équations de l'hydrodynamique et lui permet de préciser la portée de la méthode, dite de séparation des variables. La théorie des groupes peut aussi laisser prévoir l'existence des solutions de Σ possédant une propriété P. On peut, quelquefois, déterminer effectivement celles-ci; mais il arrive que, ce faisant, on trouve d'autres solutions de Σ, d'importance capitale au point de vue physique. Enfin, l'auteur insiste sur les liens étroits de la théorie des groupes avec celle de la similitude. En résumé, le concept de groupe apparalt comme un puissant principe d'unification en mécanique des fluides, comme c'est le cas d'autres disciplines physico-mathématiques.

Au chapitre V, l'auteur entroduit la notion classique de masse virtuelle (ou induite) d'un solide immergé dans un fluide. Après avoir attaché à cette masse un tenseur et étudié les propriétés de celui-ci, l'auteur passe en revue les applications pratiques de la notion en cause (tunnels aérodynamiques, période de vibration des navires, etc.). Il parvient aussi par ce moyen à réduire à la forme lagrangienne les équations du mouvement du corps lorsque le fluide est dépourvu de surface libre. Cela amène l'auteur à présenter quelques applications de la théorie des groupes à l'integration des équations de la dynamique classique. Ce chapitre illustre la puissance des méthodes que l'on a passé en revue dans les chapitres antérieurs.

La brève analyse qui précède aura convaincu le lecteur de l'intérêt de l'ouvrage de Birkhoff. Les sujets traités, tant classiques que nouveaux sont toujours envisagés dans la perspective des mathématiques modernes. La richesse même de la table des matières contraint trop souvent l'auteur à se limiter à des brèves indications. Aussi, ce petit volume ne parait pas destiné à un néophyte. Mais il contribuera à faire connaître quelques théories modernes et capitales de l'hydrodynamique théorique; il inspirera aux chercheurs des réflexions fructueuses, parfois des objections, sans jamais laisser le lecteur indifférent.

J. Kravtchenko et R. Gerber (Grenoble).

Ertel, Hans, and Rossby, Carl-Gustaf. A new conservation theorem of hydrodynamics. Geofis. Pura Appl. **14**, 189–193 (1949).

The theorem in question is that announced and proved in an earlier paper [S.-B. Deutsch. Akad. Wiss. Berlin. Math.-Nat. Kl. **1949**, no. 1; these Rev. **12**, 58]. Two proofs are given. The first, using the "Lagrangian" equations of motion, is that given in the former paper. The second consists in putting the Hamilton principal function for the arbitrary function which occurs in the form of Helmholtz's vorticity equation given earlier by Ertel [Meteorol. Z. **59**, 277–281 (1942)]. *C. Truesdell* (Bloomington, Ind.).

Rose, Alan. On the use of a complex (quaternion) velocity potential in three dimensions. Comment. Math. Helv. **24**, 135–148 (1950).

This paper approaches axially symmetric flows by a method paralleling complex function theory in the hydrodynamics of plane flows; it uses as the appropriate tool the theory of analytic functions of a quaternion [see Fueter, same journal **7**, 307–330 (1935)]. For a given flow the author defines a new stream function $\psi(x, y, z, \xi, \eta, \zeta)$ of the pair of points $A(x, y, z)$, $B(x+\xi, y+\eta, z+\zeta)$ to be the rate of flow of fluid across the plane triangle formed by these two points and the origin (the sign of ψ being based on a suitable convention). If the fluid is incompressible and the flow axially symmetric and irrotational, with φ the potential, and if we define $(\psi_1, \psi_2, \psi_3)=(\partial\psi/\partial\xi, \partial\psi/\partial\eta, \partial\psi/\partial\zeta)_{\xi,\eta,\zeta=0}$, then the quaternion function $\varphi+i\psi_1+j\psi_2+k\psi_3$ satisfies the conditions for it to be a right-regular function of the quaternion $w+ix+jy+kz$, where i, j, k are the usual quaternion units and w is an imaginary fourth coordinate whose axis is perpendicular to the other three. Thus the theory of analytic quaternion functions is in principle applicable. The author considers examples of several specific flows for which he determines the stream function explicitly, and, in particular, he finds the complex quaternion potential for the uniform flow past a sphere by a method entirely analogous to the complex variable method for the cylinder.

D. Gilbarg (Bloomington, Ind.).

Goldstein, S., and Lighthill, M. J. A note on the hodograph transformation for the two-dimensional vortex flow of an incompressible fluid. Quart. J. Mech. Appl. Math. **3**, 297–302 (1950).

The authors state: "In order to demonstrate the occurrence of branch lines in the hodograph transformation of the two-dimensional vortex flow of an incompressible inviscid fluid [branch lines are curves in the hodograph plane along which the Jacobian of the hodograph transformation vanishes] we work out the hodograph plane corresponding to the flow of a stream with uniform shear past a circular cylinder. This is a simple mathematical example, of no physical interest in itself, intended purely to exhibit the kind of mathematical difficulty that may arise in using the hodograph transformation for other problems involving vortex motion." Several of the general results stated by Craggs [Proc. Cambridge Philos. Soc. **44**, 360–379 (1948); these Rev. **10**, 640] in his study of the geometry of the hodograph transformation, e.g., the folding back (in general) of the hodograph plane along branch lines, and the fact that the branch lines are usually cusped in the hodograph plane, are verified in the particular example of this paper, in which the hodograph plane proves to be a Riemann surface of six sheets. *D. Gilbarg* (Bloomington, Ind.).

Dolaptchiev, Bl. Stabilisation des files de tourbillons. C. R. Acad. Bulgare Sci. Math. Nat. **2**, no. 2-3, 13–16 (1949).

The author considers two infinite parallel rows of equally spaced point vortices, calling the general configuration of this type "asymmetric" to distinguish it from the well-known symmetric and alternating vortex streets. The asymmetric configuration, which is known to be in equilibrium, has been shown by Godefroy [Comment. Math. Helv. **11**, 293–320 (1939)], Maue [Z. Angew. Math. Mech. **20**, 129–137 (1940); these Rev. **2**, 170], and the author, to be stable (with respect to a certain wide class of small perturbations), provided the configuration satisfies the relation (*) $\sinh h\pi/l=\sin d\pi/l$, where h is the distance between the rows, l the spacing of the vortices in each row, and d the distance along the axis that each row is displaced from the symmetrical configuration. The author shows that if a stable Karman vortex street is perturbed through a sequence of asymmetric configurations, the velocity of the system being that of the instantaneous asymmetric configuration, then the motion of the vortices is such that (*) is satisfied at each instant. The author uses this result as basis for a qualitative explanation of why the Karman street is the only stable configuration observed in nature. *D. Gilbarg.*

Viguier, Gabriel. Structure analytique de la nouvelle mécanique des fluides visqueux. Revue Sci. **87**, 86–88 (1949).

The author makes a formal extension of the theory of viscous fluids by assuming that the viscous stresses depend on velocity derivatives in a nonlinear manner, including terms of the third order. *C. C. Lin* (Cambridge, Mass.).

Aržanyh, I. S. The integral equations of steady motion of a viscous incompressible fluid. Doklady Akad. Nauk SSSR (N.S.) **74**, 21–24 (1950). (Russian)

The author considers the following two problems for viscous incompressible fluids: (1) the motion of a fluid inside a closed rigid container rotating with constant angular velocity; (2) the motion of a fluid outside a rigid body

moving with constant velocity, the fluid being at rest at infinity. The author shows that the absolute velocity and the pressure can be expressed in terms of quantities determined by seven (two vectors and a scalar) integral equations which are nonlinear inasmuch as the vector product of the two unknown vector functions enters. Simplifications for slow motion and the possibility of solving the equations by a series expansion are discussed briefly.

J. V. Wehausen (Providence, R. I.).

Gandin, L. S. On the convergence of the method of Švec. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 441–443 (1950). (Russian)

The polynomial approximation for the boundary layer velocity profile was developed by Švec [same journal **13**, 257–266 (1949); these Rev. **11**, 277] by application of a method of iteration. By organizing in the same manner a rigorous solution of the problem, the convergence of the iterative process is established.

N. A. Hall.

Whitehead, L. G., and Canetti, G. S. The laminar boundary layer on solids of revolution. Philos. Mag. (7) **41**, 988–1000 (1950).

By taking the velocity potential α and the Stokes stream function β as the independent variables, the authors obtain as the equation for the boundary layer flow in a solid of revolution:

$$q\frac{\partial q}{\partial \alpha}+wy_0\frac{\partial q}{\partial \beta}=-\frac{1}{\rho U^2}\frac{\partial p}{\partial \alpha}(1-q^2)+\frac{y_0^2}{Re}\frac{\partial^2 q}{\partial \beta^2}.$$

Here q and w are the velocity ratios; p, ρ, and U pressure, density, and undisturbed velocity, respectively; y_0 the distance of the boundary from the axis of symmetry; and Re the Reynolds number. In the case of a cone and a rounded-nose body, the solutions are shown to be reducible to those of Hartree for flows past a wedge [Proc. Cambridge Philos. Soc. **33**, 223–239 (1937)]. The approximate solution by an integral method, which is exact in the vicinity of the stagnation point, is compared with accurate solutions for flow past a cone, and good agreement is obtained. As a final example the flow past an ellipsoid is considered. Satisfactory results are found over the forward 80% of the boundary layer. The method fails, however, near the separation point.

Y. H. Kuo (Ithaca, N. Y.).

Foote, J. R., and Lin, C. C. Some recent investigations in the theory of hydrodynamic stability. Quart. Appl. Math. **8**, 265–280 (1950).

Dans les recherches théoriques sur la stabilité des écoulements visqueux à deux dimensions, le fluide en mouvement est généralement supposé borné par des parois solides. Le présent travail est consacré à l'étude de quelques aspects de ce problème de la stabilité quand la frontière est constituée par des lignes libres. Ces recherches intéressent donc en particulier les jets et les sillages plans. Les auteurs étudient spécialement la nature des solutions asymptotiques de l'équation de stabilité et étendent les travaux de Wasow [Ann. of Math. (2) **49**, 852–871 (1948); ces Rev. **10**, 377] sur la validité de ces solutions. L'application de ces résultats aux sillages et aux jets montre que les solutions "visqueuses" doivent être écartées et que l'effet de la viscosité intervient dans les approximations du second ordre des solutions "non visqueuses." Nous signalerons également un théorème d'après lequel l'existence d'un extrêmum du tourbillon est une condition nécessaire et suffisante à l'existence de perturbations. Par ailleurs l'étude des forces de viscosité indique que dans certains cas les perturbations peuvent avoir pour effet de transmettre une quantité de mouvement d'une région à une autre région de vitesse moyenne plus grande. La dernière partie du travail est consacrée à une application de la théorie envisagée à l'étude de la stabilité des couches zonales de vent sur la sphère terrestre.

R. Gerber.

Chandrasekhar, S. The theory of axisymmetric turbulence. Philos. Trans. Roy. Soc. London. Ser. A. **242**, 557–577 (1950).

The author develops the theory of axially symmetric tensors and their application in the statistical theory of turbulence by the method used by H. P. Robertson for isotropic turbulence [Proc. Cambridge Philos. Soc. **36**, 209–223 (1940); these Rev. **1**, 286]. Although Batchelor [Proc. Roy. Soc. London. Ser. A. **186**, 480–502 (1946); these Rev. **8**, 238] has also considered this problem, the author carries it further to obtain an analogue of the Kármán-Howarth equation of isotropic turbulence.

Let $\boldsymbol{\lambda}$ be the preferred direction, $\boldsymbol{\xi}=\mathbf{r}-\mathbf{r}'$, $(\boldsymbol{\xi}, \boldsymbol{\xi})=r^2$, and $(\boldsymbol{\xi}, \boldsymbol{\lambda})=r\mu$. A solenoidal axisymmetric tensor Q_{ij}, symmetric in the indices i, j, can be written in the form

$$Q_{ij}=A\xi_i\xi_j+B\delta_{ij}+C\lambda_i\lambda_j+D(\lambda_i\xi_j+\xi_i\lambda_j),$$

where A, B, C, D depend upon two scalar functions of r and μ, Q_1 and Q_2, the "defining scalars" of Q_{ij}. If $Q_{ij}=\overline{u_iu_j'}$, where $\mathbf{u}$ is the velocity at $\mathbf{r}$ and $\mathbf{u}'$ that at $\mathbf{r}'$, then the analogue of the Kármán-Howarth equation is the pair of equations

$$\partial Q_1/\partial t=2\nu\Delta Q_1+S_1,$$
$$\partial Q_2/\partial t=2\nu\Delta Q_2+4\nu r^{-2}\partial^2 Q_1/\partial\mu^2+S_2,$$

where $\Delta=\partial^2/\partial r^2+4r^{-1}\partial/\partial r+(1-\mu^2)r^{-2}\partial^2/\partial\mu^2-4\mu r^{-2}\partial/\partial\mu$ and S_1, S_2 are the defining scalars for the tensor

$$S_{ij}=\partial(\overline{u_iu_ku_j'}-\overline{u_iu_k'u_j'})/\partial\xi_k+\rho^{-1}(\partial\overline{pu_j'}/\partial\xi_i-\partial\overline{p'u_i}/\partial\xi_j).$$

J. V. Wehausen (Providence, R. I.).

Chandrasekhar, S. The decay of axisymmetric turbulence. Proc. Roy. Soc. London. Ser. A. **203**, 358–364 (1950).

The equations derived in the paper reviewed above are applied to the study of the decay of axisymmetric turbulence when the inertial terms, i.e., S_1 and S_2, may be neglected ("the last stages of decay"; cf. Batchelor and Townsend [same Proc. Ser. A. **194**, 527–543 (1948); these Rev. **10**, 339]). The equation for $Q_1(r, \mu; t)$, for given initial condition $Q_1(r, \mu; 0)$, may then be solved, giving Q_1 expanded in a series of Gegenbauer polynomials $C_{2n}^{3/2}(\mu)$. The function Q_2 may then be found. The asymptotic behavior as $t\rightarrow\infty$ is investigated and it is shown that

$$Q_i(r, \mu; t)\rightarrow -\Lambda_i e^{-r^2/8\nu t}/48(2\pi)^{1/2}(\nu t)^{5/2},$$

where $\Lambda_i=-\int_0^\infty q_0^{(i)}(r)r^4\,dr$ and where $q_0^{(i)}(r)$ is the first coefficient in the expansion $Q_i(r, \mu; 0)=\sum_0^\infty q_{2n}^{(i)}(r)C_{2n}^{3/2}(\mu)$. Asymptotic values of various other quantities of interest are also given. Finally, the author also considers a particular case of axisymmetric turbulence in which Q_1 is independent of μ. It is shown that then Q_1 satisfies an equation identical with the Kármán-Howarth equation and also has the Loitsyansky invariant; thus, in the particular case, the turbulence behaves like a superposition of two noninteracting fields of turbulence, one isotropic, one axisymmetric.

J. V. Wehausen (Providence, R. I.).

Schlichting, H. Turbulence and heat stratification. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1262**, 55 pp. (1950).

Translated from Z. Angew. Math. Mech. **15**, 313–338 (1935).

Flax, Alexander H. On a variational principle in lifting-line theory. J. Aeronaut. Sci. 17, 596–597 (1950).

A variational problem is formulated, the solution of which is the same as the solution of the integro-differential equation of lifting-line theory in aerodynamics. It is stated that this variational formulation can be extended to more general cases, involving spanwise nonuniform velocities and involving wing-body interaction. *E. Reissner.*

Legras, M. J. Calcul numérique de la répartition de la circulation sur une aile de flèche accentuée. Recherche Aéronautique **1950**, no. 16, 39–53 (1950).

The author generalizes the subsonic wing theory of Prandtl to the case of sweptback wings. The resulting formulas are sensitive to round-off errors of calculation, and a primary aim of the paper is to transform them to a form suitable for numerical calculation. In addition, the formulas are modified in accordance with experimental results to take care of aspect ratio effects. A very complete outline and tabulation of calculations is given. The pressure distribution so calculated compares favorably with experimental results on a 45° sweptback wing. *D. P. Ling.*

Miles, John W. Quasi-stationary airfoil theory in subsonic compressible flow. Quart. Appl. Math. **8**, 351–358 (1951).

A solution of the integral equation for an oscillating, two-dimensional, thin airfoil in a compressible flow (subsonic and inviscid) is obtained by retaining only first order terms in frequency. The results are applied to the calculation of the damping derivative of a tail in rotary motion about a forward center, and it is shown that the damping is considerably less than that calculated on the basis of stationary airfoil theory. A brief investigation of induction effects shows this reduction to be considerably less for a wing of finite aspect ratio. *Author's summary.*

Miles, John W. Transient loading of supersonic rectangular airfoils. J. Aeronaut. Sci. 17, 647–652 (1950).

The results of earlier papers [cf., e.g., same J. 15, 592–598 (1948); these Rev. 10, 411] are used to calculate the lift and moment coefficients for a rectangular airfoil that experiences a sudden change in angle of attack (without rotation) in an otherwise uniform supersonic stream or which enters a sharp-edged gust. The results are valid when the Mach lines from the leading-edge corners do not intersect the opposite side edges. Numerical results are given in the form of curves. *Author's summary.*

Legendre, Robert. Sur certaines solutions des équations de l'écoulement plan d'un fluide pour une loi de compressibilité approximative. C. R. Acad. Sci. Paris **231**, 1419–1421 (1950).

Cherry, T. M. Exact solutions for flow of a perfect gas in a two-dimensional Laval nozzle. Proc. Roy. Soc. London. Ser. A. **203**, 551–571 (1950).

It is known that the solution of an isentropic compressible flow through a convergent-divergent channel, which is subsonic and supersonic in respectively the convergent and divergent sections, becomes three-valued in the supersonic region when mapped into the hodograph plane. To deal with this difficulty, the author first expresses the hypergeometric functions in terms of Bessel functions and then transforms part of the solution into Kapteyn series. By the known properties of a summed Kapteyn series, the author is able to construct a solution possessing all the desired behaviors. Since this series is singular on the branch lines, the solution is continued to the rest of the region by a number of steps, each one of which is restricted to a specific but overlapping domain. *Y. H. Kuo* (Ithaca, N. Y.).

Mack, Charles E., Jr., and Kolodner, Ignace I. Linearized treatment of supersonic flow through axi-symmetric ducts with prescribed wall contours. S.M.F. Fund Paper No. **286**. Institute of the Aeronautical Sciences, New York, N. Y., 1950. 63 pp. $1.25.

This is an application of the linear small perturbation theory of supersonic flow to problems of the internal steady axisymmetric flow through a duct of nearly constant cross-section. First, a numerical method of solution is given, based on determining a distribution of source rings, on a circular cylinder through the lip of the duct, such as will cause the boundary condition at the surface to be satisfied. There is a considerable discussion of the fact that if the duct becomes parallel the flow continues to oscillate without damping. Secondly, a discussion of the problem based on the Laplace transformation is given, in which solutions in the form of Fourier-Bessel expansions are approximated by replacing the functions of the Fourier-Bessel system by their asymptotic forms. This approximate solution shows up clearly the nature of the oscillations. In this second part the authors seem to follow G. N. Ward [Quart. J. Mech. Appl. Math. 1, 225–245 (1948); these Rev. 10, 77] very closely. *M. J. Lighthill* (Manchester).

Kaleckaya, È. M. On the theory of hydraulic shock in gas conduits. Doklady Akad. Nauk SSSR (N.S.) **72**, 1029–1032 (1950). (Russian)

The author reduces the problem considered to the nonlinear partial differential equation

$$(1) \qquad Q_{xx}=k^{-1}\{[k/(k-1)]\log Q+(k-1)AQ^{k/(k-1)}\}_{tt}.$$

Here the unknown function $Q(x, t)$ is the specific energy connected with the pressure p and density ρ by the equations $\rho Q_x=p_x$, $\rho Q_t=p_t$, k the adiabatic exponent, and A a (small) constant depending on k, the initial radius R_0, thickness δ, and elastic modulus E of the pipe:

$$A=(2R_0/\delta E)[(k-1)/kC]^{1/(k-1)}$$

(C being the constant in the adiabatic relation $p=C\rho^k$). Two approximate integration methods are proposed. (I) Set $Q=c[1+Q_1(x, t)]$, c a constant, and expand the terms in the right hand side of (1) into Taylor series in Q_1 and neglect powers higher than the first. This leads to the wave equation for Q_1. (II) Set $A=0$ and solve the equation $Q_{xx}=(k-1)^{-1}[\log Q]_{tt}$ by separation of variables ($Q=X(x)T(t)$). This leads to explicit solutions involving a certain number of arbitrary constants. *L. Bers.*

Thomas, T. Y. The determination of pressure on curved bodies behind shocks. Comm. Pure Appl. Math. **3**, 103–132 (1950).

A pointed symmetrical body is placed in a uniform two-dimensional flow. The attached bow wave is assumed to be curved, so that rotational flow holds between it and the body contour. The arbitrary zero of entropy S is chosen so that this contour is the streamline $S=0$. The pressure near the body $p(\omega, S)$ (ω=flow inclination) is expanded in a power series in S. It is shown that conditions at the shock and at the body contour lead to an ordinary differential equation for $p(\omega, 0)$, the pressure along the body. This

equation is of infinite order. Its partial sums furnish a series of approximants which are investigated as far as the first three terms. *D. P. Ling* (Murray Hill, N. J.).

Lighthill, M. J. The diffraction of blast. II. Proc. Roy. Soc. London. Ser. A. **200**, 554–565 (1950).

The present paper is a continuation of the work begun in an earlier paper [same Proc. Ser. A. **198**, 454–470 (1949); these Rev. **11**, 478]. The earlier work concerned diffraction of a blast wave, at nearly glancing incidence on a plane wall, when a corner was encountered. The present paper deals with the reflection and diffraction of blast under nearly head-on incidence with a corner. The corner in each case represents the meet of two planes under an angle close to 180°.

The mathematical methods are very similar to those employed in part I. The results can be summarized as follows: In the case of a corner convex to the shock, the incident shock travels away from the corner in either direction, and in the neighborhood of contact the reflected shock is plane, and is in fact that determined by the theory of regular reflection. The flow behind it is in consequence uniform. In the vicinity of the corner these properties break down. Within a circle whose center is at the corner, and whose radius grows with the local velocity of sound, the shock is curved, the flow is nonuniform, and the over-pressure is relatively weak, owing to the scattering effect of the corner. When the corner is concave to the blast, regular reflection holds until the incident shock has reached the corner itself. Following this, the reflected shock near the corner is curved within a circle, as before; and this circle marks the path of two subsidiary shocks not present in the previous case. Evidence is given to support the contention that much the same state of affairs must exist, even quantitatively, for considerably larger corner angles than those considered here. *D. P. Ling* (Murray Hill, N. J.).

Zeldovich, Y. B. On the theory of the propagation of detonation in gaseous systems. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1261**, 50 pp. (1950).

Translated from Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **10**, 542–568 (1940).

Davies, D. R. A note on three-dimensional turbulence and evaporation in the lower atmosphere. Proc. Roy. Soc. London. Ser. A. **202**, 96–103 (1950).

Suite d'un travail du même auteur [Quart. J. Mech. Appl. Math. **3**, 51–63 (1950); ces Rev. **12**, 141]. L'auteur a simplifié la solution et a réussi à exprimer la distribution théorique de la vapeur en utilisant les fonctions incomplètes Γ (tabulées par Pearson), ce qui réduit énormément le calcul numérique. Déjà, le premier terme de ces séries, pour l'axe central de l'aire, donne le résultat de la théorie bi-dimensionnelle. On trouve des résultats en accord assez satisfaisant avec les expériences. *M. Kiveliovitch.*

Doporto, M. Theory and description of a gradient wind computer. Meteorol. Service. Geophys. Publ., Dublin **3**, no. 1, 8 pp. (1950).

A quadratic equation is derived for the gradient wind as a function of the speed of the pressure system, the geostrophic wind speed of the radius of curvature of the isobars, and the wind direction relative to the direction of motion of the center of the pressure system. The isobars are assumed to coincide with the streamlines and the pressure systems are presumed to move without change. A gradient wind computer is described which leads to a rapid graphical solution of this equation. *H. Panofsky* (New York, N. Y.).

Harkevič, A. A. On the construction of qualitative diffraction charts. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **19**, 822–827 (1949). (Russian)

Considering the impact upon an obstacle of an acoustic wave moving with velocity c, the author derives by physical arguments a principle to the effect that $\int p\,dV$ (p the excess pressure, dV the volume element), taken over the field reached by the wave, depends only on the source and not on the obstacle. For certain cases involving symmetry it is maintained that the principle applies to any pair of symmetrically placed volume elements, this leading to certain symmetry properties of the field. The resulting information is regarded as supplementary to charts illustrating the progress of the wave-fronts according to Huygen's principle; such charts are drawn for particular cases of diffraction by screens with apertures. *F. V. Atkinson* (Ibadan).

Harkevič, A. A. Nonstationary problem of diffraction of a plane wave from a rectilinear boundary. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **19**, 828–832 (1949). (Russian)

A plane pressure wave (unit potential behind the wave-front, zero potential before it) is propagated with velocity c over a rigid half-plane screen whose edge is parallel to the wave-front. The problem is to determine the pressure at points in the diffraction zone, which in the author's theory is an expanding cylindrical region, axis the edge of the half-plane and radius ct. By means of the transformation $\cosh s = ct/r$ he finds a solution in the form of a Fourier series. He considers separately the cases in which the wave moves (i) parallel to the half-plane, (ii) perpendicular to it, and (iii) at an arbitrary angle. In the first two of these cases the series is summed explicitly and diagrams are drawn illustrating equipotentials. It is possible to apply to the problem the ideas of the paper reviewed above. This and the paper reviewed below overlap with a later work of the author [Doklady Akad. Nauk SSSR (N.S.) **72**, 45–47 (1950); these Rev. **11**, 755]. *F. V. Atkinson* (Ibadan).

Harkevič, A. A. Nonstationary radiation from a half-plane. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **19**, 833–838 (1949). (Russian)

The author [see the preceding review] now sets up boundary problems for the cylindrical diffraction region associated with impulsive motion of the plane or with uniform source distributions on one or both sides of the plane. Following the method of a previous [unavailable] paper, he derives solutions in the form of Fourier series, which he sums. Diagrams are drawn for the isobars. The reactive force on an area of the half-plane is also considered. In an appendix, similar results are obtained by a direct method of quadratures, the integrals apparently involving discontinuities of delta-function type. *F. V. Atkinson.*

Elasticity, Plasticity

Bloh, V. I. Stress functions in the theory of elasticity. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 415–422 (1950). (Russian)

Finzi [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **19**, 578–584 (1934)] indicated that the general

solution of the equilibrium equations for continuous media is $t^{ij}=\epsilon^{ikr}\epsilon^{jhs}\chi_{rs,kh}$, where χ_{rs} is an arbitrary symmetric tensor determinate only to within the quantity $v_{r,s}+v_{s,r}$, where v_r is an arbitrary vector. The author also claims discovery of this result, referring to an earlier paper [Trudy konferencii po optičeskomu metodu izučeniya napryaženiĭ, ONTI, 1937, pp. 69–73]. He then states the numbers of essentially different forms of the result in which by choice of v_r it is possible to reduce the number of nonzero components of $\chi_{rs}+v_{h,s}+v_{s,h}$ to three. For rectangular Cartesian coordinates there are five such forms, those of Maxwell and Morera being included; in general coordinates, twenty; in cylindrical coordinates, eighteen; in cylindrical coordinates for problems with axial symmetry, ten; in spherical coordinates, nineteen. The author then writes down the form taken by the Beltrami-Michell equations of classical elasticity when expressed in terms of the stress functions χ_{rs}, a result he had given earlier [loc. cit.]. He finds the general solution of these equations in terms of arbitrary functions and then gives an expression for the displacement in terms of the stress functions. Finally he gives a mechanical interpretation for the stress functions [cf. Peretti, Atti Sem. Mat. Fis. Univ. Modena **3**, 77–82 (1949); these Rev. **11**, 557]. *C. Truesdell.*

Filin, A. P. On a consequence of a variational principle of the theory of elasticity. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 451–452 (1950). (Russian)

It is shown that the principle of minimum complementary energy (Castigliano's principle) also represents the continuity condition for an arbitrarily stressed body.

H. I. Ansoff (Santa Monica, Calif.).

Fridman, M. M. The mathematical theory of elasticity of anisotropic media. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 321–340 (1950). (Russian)

This paper is a survey of the development, in Russia, of the theory of an anisotropic elastic body. The first section deals with the formulation of the plane problem, and the existence and uniqueness of solutions. The second section is devoted to effective methods of solution of plane problems. The third section contains a discussion of torsion and bending of anisotropic prismatic beams, and symmetric deformation of anisotropic bodies of revolution. A bibliography of 88 articles concludes the paper. *J. B. Diaz.*

Sekiya, Tuyosi, and Saito, Atusi. Some applications of Dirac's δ-function on the mathematical theory of elasticity. J. Osaka Inst. Sci. Tech. Part I. **1**, 99–110 (1949).

Položiĭ, G. N. Solution of the third fundamental problem of the plane theory of elasticity for an arbitrary finite convex polygon. Doklady Akad. Nauk SSSR (N.S.) **73**, 49–52 (1950). (Russian)

The third fundamental boundary value problem of plane elasticity, in the author's terminology, is the problem of finding the stresses and displacements in a plane domain G whose boundary is L when the normal displacement and the tangential stress are prescribed on L. An explicit solution of this problem is given in the particular case when L is a bounded convex plane polygon and G is its interior. The solution is based on certain formulas given earlier for the case when the boundary of G is piecewise linear [same Doklady (N.S.) **66**, 177–180 (1949); Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 297–306 (1949); these Rev. **11**, 68, 285]. Using Goursat's formula [Bull. Soc. Math. France **26**, 236–237 (1898)] $\varphi(z)+\bar{z}\psi(z)$, $z=x+iy$, for an arbitrary complex-valued solution of the biharmonic equation, where φ and ψ are analytic functions, the problem reduces to the determination of the analytic functions φ and ψ. The function φ is obtained at once from the boundary data, and ψ is obtained as the solution of a function theoretic boundary value problem of the type solved in Mushelišvili [Singular Integral Equations . . . , OGIZ, Moscow-Leningrad, 1946, p. 279; these Rev. **8**, 586]. *J. B. Diaz.*

Rothman, M. Isolated force problems in two-dimensional elasticity. I. Quart. J. Mech. Appl. Math. **3**, 279–296 (1950).

The problems of plane strain and of generalized plane stress in thin plates have been reduced to the determination of two complex functions by Stevenson [Proc. Roy. Soc. London. Ser. A. **184**, 129–179 (1945); these Rev. **8**, 115], and by N. I. Mushelišvili [I. S. Sokolnikoff, Mathematical theory of elasticity, mimeographed lecture notes, Brown University, Providence, R. I., 1941, pp. 243–318]. In the present paper the author works out a number of problems, using Stevenson's theory. First, a determination is made of the types of singularity of the two complex functions $\Omega(z)$ and $\omega(z)$ at points of concentrated loading by forces and by couples. The functions $\Omega(z)$ and $\omega(z)$ are then determined in the case of a thin infinite plate containing a single hole which is either circular, or elliptical, or is a certain curvilinear polygon which is approximately circular; in each case a number of concentrated forces act on the boundary of the hole.

G. E. Hay (Ann Arbor, Mich.).

Udoguchi, Teruyoshi. Analysis of centrifugal stress in a rotating disc containing an eccentric circular hole. Jap. Sci. Rev. Ser. I. **1**, no. 1, 53–63 (1949).

The problem described in the title is considered as one of plane stress in bipolar coordinates. The author starts with the solution for the concentrically rotating solid disk and then proceeds with the problem of removing the tractions from the boundary of an eccentric circular hole. The tractions are separated into two parts, one of which is self-equilibrating and the other of which has a resultant equal to the centrifugal force of the matter removed from the hole. The former is annulled by employing Jeffery's series solution in bipolar coordinates. The latter is first made self-equilibrating through the use of Mindlin's solution for a disk containing eccentric forces. The remaining tractions are removed only approximately, but it is shown that the residual tractions are small. The author apparently was not aware of a solution by the reviewer for an eccentrically rotating disk [Philos. Mag. (7) **26**, 713–719 (1938)]. If this had been used in place of the reviewer's solution for eccentric forces, the author's solution would not have been as lengthy. The paper concludes with results of stress computations and it is observed that the stress in the neighborhood of the hole is similar to that for a semi-infinite plate, containing a circular hole, under a uniform tension parallel to the straight edge. The latter problem was considered by Jeffery and an error in it was noted and corrected by the author [Trans. Jap. Soc. Mech. Engrs. **13**, 17–40 (1947)] and the reviewer [Proc. Amer. Soc. Civil Engrs. **65**, 619–642 (1939); Proc. Soc. Experimental Stress Analysis **5**, no. 2, 56–68 (1948)].

R. D. Mindlin (New York, N. Y.).

Kiltchevsky, N. Les méthodes approchées pour déterminer les déplacements dans les enveloppes cylindriques. Akad. Nauk Ukrain. RSR. Zbirnik Prac' Inst. Mat. **1946**, no. 8, 97–110 (1947). (Ukrainian. Russian and French summaries)

The author shows a method of reducing the problem of elastic equilibrium of cylindrical shells to the solution of a system of integral equations, by applying the Betti theorem. In his previous publications on the subject the author used the same method, but in this one he introduces an auxiliary system of displacements in a plate. He uses a special coordinate system through the middle surface of the shell, which allows him to establish a one-to-one correspondence between the points of the plate and the points of the shell. Using the Betti theorem the solution of the elastic equilibrium of the shell consists of the sum of two quantities: the solution of the elastic equilibrium of the plate and an expression which depends on the curvature of the shell. The integral equations or, in the case of free boundaries, the integro-differential equations can be solved approximately by iteration, assuming that the curvature of the shell is sufficiently small. *T. Leser* (Lexington, Ky.).

Šapiro, G. S. Elastic equilibrium of a paraboloid of revolution. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 672–673 (1950). (Russian)

Jung, H. Ein Beitrag zur Loveschen Verschiebungsfunktion. Ing.-Arch. **18**, 178–190 (1950).

Love's displacement function is used, which satisfies the biharmonic equation and expresses the stresses and displacements in an elastic body having axial symmetry. Particular solutions involving a parameter are combined by taking the integral of the product of a particular solution and a function of the parameter. The boundary conditions then appear as integral equations which are solved in various ways, for example, by use of Hankel transforms and Fourier transforms. Problems considered are given load or deflection on a half plane, thick plate of infinite extent with surface loading, infinite cylinder with given radial loading, semi-infinite cylinder with given end displacement, and rigid body pressed against a half plane. The author seems unaware of substantially similar work by Harding and Sneddon [Proc. Cambridge Philos. Soc. **41**, 16–26 (1945); these Rev. **6**, 251], Sneddon [ibid. **42**, 260–271 (1946); these Rev. **8**, 117], and Tranter and Craggs [Philos. Mag. (7) **36**, 241–250 (1945); these Rev. **7**, 352]. *E. H. Lee* (Providence, R. I.).

Favre, Henry, et Chabloz, Eric. Étude des plaques circulaires fléchies d'épaisseur linéairement variable. Cas d'une surcharge uniformément répartie. Z. Angew. Math. Physik **1**, 317–332 (1950).

This paper deals with transverse bending of thin circular plates with thickness variation assumed in the form $h=[1+\lambda\{2(r/a)-1\}]h_0$, where r is the radial coordinate and λ, a, and h_0 are constants. The solution of the linear differential equation of the problem is developed in powers of the parameter λ. Numerical applications are made to plates with clamped and with simply supported edges. *E. Reissner* (Cambridge, Mass.).

Southwell, R. V. On the analogues relating flexure and extension of flat plates. Quart. J. Mech. Appl. Math. **3**, 257–270 (1 plate) (1950).

A review is given of earlier work on two analogues relating the bending of plates and their extension. One analogue relates the lateral displacements of a bent plate with Airy's stress-function for a plate subject to extension. The other analogue relates the tractions in a bent plate with the extensions in a stretched plate. These analogues are then combined in a single analogue suitable for problems having mixed boundary conditions. The application of the boundary conditions in the case of plates having holes is thoroughly discussed. *S. Levy* (Washington, D. C.).

Houbolt, John C., and Stowell, Elbridge Z. Critical stress of plate columns. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2163**, 16 pp. (1950).

It is found that the flexural rigidity of a thin plate loaded as a column varies from EI for a plate whose width-length ratio is 0.1 or less to $EI/(1-\mu^2)$ for a plate whose width-length ratio is 10 or more. Poisson's ratio of the material is denoted by μ. Charts are given from which the flexural rigidity can be determined for intermediate values of the width-length ratio both for plates with simply supported ends and for plates with fixed ends. For plates with simply supported ends an exact solution is obtained from the differential equation of the plate, while for plates with fixed ends an approximate solution is obtained by an energy method. *H. W. March* (Madison, Wis.).

Ashwell, D. G. The anticlastic curvature of rectangular beams and plates. J. Roy. Aeronaut. Soc. **54**, 708–715 (1950).

The author considers the pure bending of a rectangular beam with a cross-section of depth d and breadth b. When d and b are comparable, the neutral plane of the beam undergoes anticlastic curvature. When d is much smaller than b, the beam becomes a plate and the neutral plane deforms into a cylinder, except near its edges. In the present paper, the distortions of a rectangular beam of any dimensions are determined, and the transition between the two types of bending is investigated. When d is much smaller than b, the material between two adjacent cross-sections is called the "transverse beam." Because of the anticlastic curvature of the plate, the equation of equilibrium of this transverse beam is found to be that of a simple beam on an elastic support. The general solution of this equation, with the appropriate boundary conditions, yields the desired result. The significant quantity seems to be b^2/Rd, where R is the radius of curvature of the strained central line of the original beam; when this quantity is comparable with or less than one, the usual theory involving anticlastic curvature applies; otherwise, it does not. *G. E. Hay.*

Borg, S. F. Additional interpretations of the solution of the straight beam differential equation. J. Franklin Inst. **250**, 249–256 (1950).

The differential equation of a transversely loaded beam with variable moment of inertia is solved by two methods. In the first a solution for a beam with built-in ends is obtained by use of a Green's function. In the second method a solution is obtained for a beam with any type of support at the ends by means of a superposition theorem utilizing a solution of the differential equation with its right-hand member replaced by unity. Interpretations of results obtained by both methods are discussed. *H. W. March.*

Freiberger, W., and Smith, R. C. T. The uniform flexure of an incomplete tore. Australian J. Sci. Research. Ser. A. **2**, 469–482 (1949).

The authors consider, first, the general problem of axisymmetric small strain in an isotropic elastic body. It is

not unusual in problems of elasticity to begin by expressing the displacement in terms of the harmonic functions introduced by Boussinesq and later employed by Papkovitch, Neuber, and others. The authors assert that these functions are valid only for rectangular coordinates and then present a lengthy study in cylindrical coordinates ending with expressions for the components of stress in terms of two harmonic functions. The author's initial assertion is, of course, incorrect. This may be concluded immediately by noting the invariant form of the expression of the (vector) displacement in terms of the Boussinesq functions. Additional evidence is contained in the large number of boundary value problems in curvilinear coordinates which have been solved in terms of these functions. Furthermore, in the author's results, their two harmonic functions are, in fact, none other than the Boussinesq functions; but this is not mentioned in the paper. There follows a statement of the boundary conditions for an incomplete tore under uniform flexure, but no specific problems are solved. In a first appendix to the paper the usual expressions are given, in curvilinear coordinates, for the displacements in terms of the Boussinesq functions. It is stated that the second appendix contains an exact solution for the case of a tore whose cross-section is a narrow rectangle. However, the solution given is only the well-known approximate one for plane stress, neglecting z-dependent terms. It may be verified that, with the z-dependent terms omitted, the solution does not satisfy the compatibility equations; with these terms retained, the boundary conditions on the cylindrical surfaces are not satisfied. *R. D. Mindlin.*

Yoshimura, Yoshimaru, and Uemura, Masuji. The buckling of spherical shells due to external pressure. Rep. Inst. Sci. Tech. Univ. Tokyo **3**, 316–322 (1949). (Japanese. English summary)

***Il'yušin, A. A. Plastičnost'. Čast pervaya. Uprugo-plastičeskie deformacii. [Plasticity. Part One. Elastic-Plastic Deformations].** OGIZ, Moscow-Leningrad, 1948. 376 pp.

Fundamental laws of elastic-plastic deformation. Fundamental equations of the theory of small elastic-plastic deformations. The simplest problems of the theory of plasticity. Equilibrium of plates and shells. Stability of plates and shells. The pressing in of dies and the supporting capacity of an incompressible plastic body. Dynamical problems of plasticity. *Table of contents.*

Mežlumyan, R. A. Flexure and torsion of thin-walled cylindrical shells beyond the elastic limit. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 253–264 (1950). (Russian)

A thin-walled cylindrical shell is assumed to be compressible and to obey in the plastic range a stress-strain law of the deformation with hardening. The paper is confined to a reduction of the equilibrium equations to a system of fourth order ordinary differential equations in terms of displacements. For the original formulation of the problem the reader is referred to the work of Vlasov [Thin-Walled Elastic Bars, Gosstroĭizdat, Moscow-Leningrad, 1940] and for methods of solving the plastic-elastic boundary value problem to the work of Il'yušin [see the preceding review]. *H. I. Ansoff* (Santa Monica, Calif.).

White, G. N., Jr., and Drucker, D. C. Effective stress and effective strain in relation to stress theories of plasticity. J. Appl. Phys. **21**, 1013–1021 (1950).

To correlate experiment with theory in plasticity, many stress-strain types of laws have been introduced. In the present paper, the authors consider several linear laws of the incremental type which involve a loading function and the increments of stress and strain (elastic plus plastic). All of these proposed stress-strain relations are compatible with the idea of work hardening (no work can be extracted from the material by carrying it through a cycle). Two-dimensional stress plots of three loading functions are analyzed: (1) the von Mises loading function, and (2), (3) two loading functions introduced by the authors. According to the authors, experiment indicates the necessity for considering loading functions which are considerably more complex. In another approach to the stress-strain relations, many authors have attempted to relate "effective stress" and "effective strain." The present authors show how the incremental type of stress-strain relations may be used to obtain reasonable definitions of the "effective stress" and "effective strain," provided that Bauschinger effects are not present. In fact, this last topic contains the principal contributions by the authors. Both isotropic and anisotropic theories are examined. The authors conclude that Dorn's formula for the increment of effective strain in terms of the increments of the plastic strain tensor seems to be most satisfactory. *N. Coburn* (Ann Arbor, Mich.).

Cicala, Placido. Sulle deformazioni plastiche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 583–586 (1950).

The author discusses some predictions of the slip theory of plasticity [Batdorf and Budiansky, Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1871** (1949); these Rev. **10**, 648] with particular reference to the application of this theory to the plastic buckling of plates. A strain-hardening plastic material is subjected to a gradually increasing uniaxial compressive stress; when this stress has reached a certain value σ_x, a small shearing stress $d\tau_{xy}$ is introduced together with an increment $d\sigma_x$ of the compressive stress; it is desired to find the apparent shear modulus $G'=d\tau_{xy}/d\gamma_{xy}$ relating the stress increment $d\tau_{xy}$ to the corresponding strain increment $d\gamma_{xy}$. It is shown that, for sufficiently large values of $d\sigma_x/d\tau_{xy}$, the prediction of the slip theory agrees with that of the deformation theory. This result which was first established, though in a less elegant way, by Batdorf [J. Aeronaut. Sci. **16**, 405–408 (1949)] is supplemented by the following new formula giving G' for $d\sigma_x/d\tau_{xy}=0$: $1/G'=1/G+3/2E_s-3/2E$. Here G denotes the elastic shear modulus, E Young's modulus, and E_s the secant modulus for the stress σ_x. *W. Prager* (Providence, R. I.).

Sokolovskiĭ, V. V. Plane equilibrium of a plastic wedge. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 391–404 (1950). (Russian)

The author solves three problems of plastic deformation of a plane wedge with linear hardening using the following boundary conditions: (a) concentrated force applied at the apex; (b) a moment applied at the apex; (c) a uniformly distributed load along one face. Problem (a) is solved in closed form and problems (b) and (c) are reduced to a numerical integration of a system of nonlinear ordinary differential equations. The author also solves the plastic-elastic problem of a plane half-space without hardening

partly loaded with a uniformly distributed pressure. For the limiting value of the pressure $p=K(\pi+2)$ the half-space becomes completely plastic and the results reduce to the well-known solution. *H. I. Ansoff.*

Belaenko, F. A. The stresses around a circular shaft in an elastic-plastic soil. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1950**, 914–925 (1950). (Russian)

Consider a deep vertical circular shaft in a homogeneous isotropic soil bounded by a horizontal surface and exhibiting linear hardening in the plastic range. Mathematically the problem is equivalent to a rotationally symmetric half-space with an infinitely long hole and a constant body force normal to the free surface. The determination of stresses is reduced to an integration of a second order ordinary nonlinear differential equation. An approximate solution is given which satisfies the differential equation at the surface of the shaft as well as the boundary conditions. The vertical stress is a function only of the distance from the free surface and the radial and the hoop stresses vary exponentially with the radial distance from the center of the shaft.

H. I. Ansoff (Santa Monica, Calif.).

Malvern, L. E. The propagation of longitudinal waves of plastic deformation in a bar of material exhibiting a strain-rate effect. Graduate Division of Applied Mathematics, Brown University, Providence, R. I., Tech. Rep. **A11-39**, ii+74 pp. (1949).

Strain-rate effects are considered in the problem stated as a possible means of accounting for the discrepancies between experiment and the theory based on quasi-static stress-strain relations [cf. Donnell, Trans. A.S.M.E. **52**, Appl. Mech., 153–167 (1930); White and Griffis, NDRC report **A-71** (OSRD no. **742**) (1942); Trans. A.S.M.E. **69**, A-337–A-343 (1947); von Kármán, NDRC report **A-29** (OSRD no. **365**) (1942)]. Experiments on tension impact on a bar have shown that a higher stress occurs at the fixed end than the quasi-static theory predicts and that the maximum strain developed was somewhat less than the static theory predicted.

If ϵ_y denotes the strain at initial yield, $\sigma=f(\epsilon)$ the quasi-static stress-strain relation, and E_0 Young's modulus, the general form of the stress-strain law used by the author is $E_0\dot{\epsilon}=\dot{\sigma}+g(\sigma,\epsilon)$ for $\sigma>f(\epsilon)$ and $\epsilon>\epsilon_y$, and $E_0\dot{\epsilon}=\dot{\sigma}$ for $\sigma\leqq f(\epsilon)$ or $\epsilon\leqq\epsilon_y$. Since the application of load in an impact problem is regarded as instantaneous, these relations imply that the instantaneous reaction is entirely elastic but that subsequently the elastic strain may decrease as the plastic strain increases. The propagation of a longitudinal loading wave of plastic deformation in a bar is described by a hyperbolic system of three first order, quasi-linear, partial differential equations expressing the equation of motion, the condition of compatibility, and the stress-strain law. This system can be integrated numerically by the method of characteristics under appropriate boundary conditions. The characteristics and the conditions which hold along them are developed.

A semi-infinite rod, instantaneously set in motion by an end velocity which is then kept constant, is first studied. An elastic shock wave is immediately formed. The solution is then determined by a step-by-step numerical integration following the characteristics in the time-displacement plane. If the impact is of finite duration, a wave of unloading must also be considered. This will be a shock wave if the stress reduction is sudden. It is shown that, in general, such a wave will be absorbed. The problem is further complicated by the determination of an elastic-plastic unloading boundary. Reflections of plastic waves in a bar of finite length are also discussed as well as the problem of unloading in such a case.

Two numerical examples are considered: instantaneous application of impact end velocity and impact end velocity rising exponentially. It is shown for an idealized law without strain-hardening that the equations can be reduced to a telegraph equation and an explicit solution given. The numerical solutions show that strain-rate effect can account for the discrepancy in the stress-time variation at the fixed end. The theory does not predict, however, the observed uniform strain region near the impact end of the bar. The author points out that the theory might be applicable to materials like lead at higher velocities where the uniform strain region does not occur. An excellent bibliography of pertinent papers in the field of plastic wave propagation is included. *G. H. Handelman* (Pittsburgh, Pa.).

Kočetkov, A. M. The approximate solution of some problems of unsteady motion of a viscous-plastic medium. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 433–436 (1950). (Russian)

The paper presents approximate solutions for two problems of unsteady motion in a viscous-plastic medium: (1) impact of a rigid cylinder on a flat plate; (2) motion of a viscous-plastic mass around a fixed cylinder.

H. I. Ansoff (Santa Monica, Calif.).

Gaskell, R. E. The calendering of plastic materials. J. Appl. Mech. **17**, 334–336 (1950).

A mathematical solution is obtained for the rolling of a viscous and of a Bingham material. The rools are of large diameter compared with the thickness of the sheet so that the basic two-dimensional equations for slow flow between parallel plates are considered adequate. Despite the absence of elasticity it is demonstrated that the sheet remains in contact with the rolls for a considerable distance past the minimum section and so is appreciably thicker than the clearance between the rolls. *D. C. Drucker.*

Hodge, P. G., Jr. Approximate solutions of problems of plane plastic flow. J. Appl. Mech. 17, 257–264 (1950).

Discontinuous stress and velocity solutions [Prager, J. Aeronaut. Sci. **15**, 253–262 (1948); Winzer and Carrier, J. Appl. Mech. **15**, 261–264 (1948); these Rev. **9**, 546; **10** 495] are used as a computational device to obtain good approximate rigid-plastic plane strain solutions. Constant state regions separated by one or more shocks replace continuous solutions. This leads to a simple and quite accurate construction of the final configuration of an initially square network of lines for problems of large deformation and also to a reasonably good distribution of surface traction for the illustrative problems of pressing and indentation [Prandtl, Z. Angew. Math. Mech. **3**, 401–406 (1923); Hill, Lee, and Tupper, Proc. Roy. Soc. London. Ser. A. **188**, 273–289 (1947); these Rev. **8**, 358].

The following basic theorem is employed: Let a stress field be composed entirely of regions of constant state, each of which moves instantaneously as a rigid body. Let these regions be numbered 1, 2, 3, $\cdots$. At any fixed time t in the process of deformation, let $D_{ijk\cdots m}$ denote the initial position, and $\Delta_{ijk\cdots m}$ the final position of that set of points which first became plastic in region i, then passed successively through regions $j, k, \cdots, m$, and finally occupied

region n at time t. Then a set of equispaced parallel lines in $D_{i_1 i_2 \cdots m_n}$ is deformed into a set of equispaced parallel lines in $\Delta_{i_1 i_2 \cdots m_n}$.

D. C. Drucker (Providence, R. I.).

Seigel, Harold O. A theory of fracture of materials and its application to geology. Trans. Amer. Geophys. Union **31**, 611–619 (1950).

Fracture is postulated to depend upon the normal and shearing stress on the fracture surface. Increase in tensile stress and increase in shear are each supposed to increase the probability of failure. Isotropic and anisotropic materials are discussed. The mathematical conclusions which are derived using undetermined multipliers and considerable algebra are obtainable directly from Mohr's circles of stress and a curve of rupture drawn in the same space. An interesting physical conclusion by the author is: "On observing a fault to be displaced laterally along another fault or shear zone a geologist might be prone to call the displaced fault the older of the two structures, but on the basis of the above theory he would be justified in examining other evidence before drawing any conclusion as to relative ages."

D. C. Drucker (Providence, R. I.).

Huang, Kun. On the atomic theory of elasticity. Proc. Roy. Soc. London. Ser. A. **203**, 178–194 (1950).

In der Dynamik der Kristallgitter war es früher üblich spezielle Arten von Wechselwirkungen der Atome anzunehmen, so besonders zentrale Kräfte oder Wechselwirkungen zwischen orientierten Molekülen. Nach der Quantenmechanik treten jedoch sehr verschiedene Arten von Bindungen in den festen Körpern auf und deshalb wurde besonders von Born und seine Mitarbeiter versucht die Theorie der festen Körper bloss unter der Annahme einer von den Kernverschiebungen abhängenden Gitterenergie auszuarbeiten. Diese Funktion soll abgesehen von der Invarianz bei entsprechenden Translationen und Rotationen weiter nicht spezialisiert werden.

In der vorliegenden Arbeit wird zuerst am Beispiel einer linearen Kette gezeigt, dass es nicht möglich ist die Deformationsenergie in der allgemeinen Theorie komplett auszudrücken. Weiter wird in einem von Spannungen freien elastischen Medium (dessen Verhalten also durch die 21 elastischen Konstanten charakterisiert wird), die Gleichung der elastischen Wellen mit der der Gitterwellen verglichen und dabei gezeigt, dass in diesem Falle die elastischen Konstanten tatsächlich mit Hilfe der in der Gleichung der Gitterwellen stehenden Grössen (Klammerausdrücke die von den zweiten Ableitungen der erwähnten Gitterenergie abhängen) ausdrückbar sind. Im nächsten Abschnitt wird die selbe Rechnung für ein sich in einem Spannungszustande befindenden Kontinuum, dessen elastische Eigenschaften im Anhang hergeleitet werden, durchgeführt. Dabei ergibt sich, dass der mittlere Druck jetzt in die erwähnten Zusammenhänge eingeht. Zur blossen Definition der gewohnten elastischen Konstanten ist es also schon notwendig, dass die Spannungen verschwinden. Zur Behandlung solcher Fragen muss eine neue mathematische Technik ausgearbeitet werden, und dabei ergibt sich, dass nur fünf Spannungskomponenten explizite ausdrückbar sind, nämlich alle anisotropen Spannungen. Nur wenn die verschwinden, sind die elastischen Konstanten ausdrückbar. Die auftretenden Verhältnisse werden am Beispiel der Zentralkräfte erläutert.

T. Neugebauer (Budapest).

MATHEMATICAL PHYSICS

Electromagnetic Theory

Pussèt, L. A. Diffraction of a homocentric beam. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **20**, 722–728 (1950). (Russian)

The beam is derived from a single spherical wave-function $\psi=(e^{-ikR}/R)u(\theta, \varphi)$, where

$$\sin\theta(\partial/\partial\theta)(\sin\theta\,\partial u/\partial\theta)+\partial^2 u/\partial\varphi^2=0,$$

and lies entirely on one side of a plane through the vertex. The field-vectors are $\mathbf{E}=i\mathbf{N}_0$, $\mathbf{H}=(k/\omega\mu)\mathbf{M}_0$, where

$$\mathbf{N}_0=(e^{-ikR}/R)\left(0, -i\frac{\partial u}{\partial\theta}, -\frac{i}{\sin\theta}\frac{\partial u}{\partial\varphi}\right),$$

$$\mathbf{M}_0=(e^{-ikR}/R)\left(0, \frac{1}{\sin\theta}\frac{\partial u}{\partial\varphi}, -\frac{\partial u}{\partial\theta}\right)$$

[cf. Stratton, Electromagnetic Theory, McGraw-Hill, New York-London, 1941, pp. 394, 415; the author refers to a Russian translation of this work]. The transformation $x=\tan(\theta/2)\cos\varphi$, $y=\tan(\theta/2)\sin\varphi$, exhibits u as a harmonic function of x and y. This is used to show that the field is finite everywhere, except at the vertex. It is also pointed out that the field is transverse. The author then considers boundary problems for the illuminated portion of a sphere, centre at the vertex. By the theory of Neumann's problem, the field-vectors along the boundary of such a portion are more than sufficient to determine the field within. The author suggests, without formal proof, that "generally speaking" the field will be determined by specification of the light intensity and polarisation along the boundary. Finally he associates with such a beam a field which is finite everywhere; this problem derives from optical systems but is not formally posed. He uses formulae due to Kottler which are, with certain discrepancies, (30) and (31) of Stratton [loc. cit., p. 469]. The nature of this field is examined for distances large compared to the wave-length.

F. V. Atkinson (Ibadan).

Heins, Albert E. The reflection of an electromagnetic plane wave by an infinite set of plates. III. Quart. Appl. Math. **8**, 281–291 (1950).

This paper continues investigations by Carlson and the author [same Quart. **4**, 313–329 (1947); **5**, 82–88 (1947); these Rev. **8**, 422, 614] concerning the reflection and transmission of monochromatic plane electromagnetic waves incident on an infinite stack of parallel staggered perfectly conducting half-planes. The papers cited above dealt with the case when (1) there is a single propagating mode in the ducts, and (2) a single reflected wave in free space. Here (1) is retained, but it is now assumed that (3) there are two reflected waves. The incident field has its magnetic vector parallel to the edges of the plates, the case which has an acoustical analogue. By using the Wiener-Hopf integral equation method, the restrictions on the wave-length and the angles of incidence and stagger are found. The resulting expressions for the transmission coefficient and the two reflection coefficients are shown to agree with a relation connecting them which follows from the Poynting vector theorem.

E. T. Copson (St. Andrews).

Schorr, Marvin G., and Beck, Fred J., Jr. Electromagnetic field of the conical horn. J. Appl. Phys. **21**, 795–801 (1950).

The purpose of this paper is "to show that the rigorous assumed-field technique [Schelkunoff, Physical Rev. (2) **56**, 308–316 (1939)], when applied to the [finite] conical horn, yields good agreement with experiment over a range of [small] cone angles for which the radiation integrals may be satisfactorily evaluated." *C. J. Bouwkamp* (Eindhoven).

Franz, Walter. Multipolstrahlung als Eigenwertproblem. Z. Physik **127**, 363–370 (1950).

The author derives the classical expressions for the electric and magnetic multipoles in spherical polar coordinates by requiring that the magnitudes of the total angular momentum and one of its components assume eigenvalues for the appropriate solutions of Maxwell's equations. Reference is made to an earlier paper by Heitler [Proc. Cambridge Philos. Soc. **32**, 112–126 (1936)]. The relation between the author's formulae and the Debye potentials [cf. J. Meixner, Z. Naturforschung **3a**, 506–518 (1948); these Rev. **11**, 141] is mentioned. Orthogonality and completeness of the set of multipoles is proved. [Reviewer's remark: In quantum electrodynamics the radial factors of multipole fields are usually expressed in terms of Bessel functions of the first kind, which makes the multipoles all regular at the origin. However, in classical electromagnetic theory the multipoles are singular at the origin, the radial factors now being spherical Hankel functions. Though the author's proof of the orthogonality does not apply for the latter case, the partial fields are still orthogonal. In fact, integration over the radial direction is not necessary; the partial fields are already normal over any sphere concentric with the origin.]
C. J. Bouwkamp (Eindhoven).

LePage, W. R., Roys, C. S., and Seely, S. Radiation from circular current sheets. Proc. I.R.E. **38**, 1069–1072 (1950).

In terms of Fourier series and Bessel-Fourier series, the authors analyze and synthesize the distant radiation field of a system of Hertzian doublets arranged in a number of coplanar concentric circles. The doublets are continuously distributed along the periphery of either circle and perpendicular to the plane of the circles. [In the reviewer's opinion, the title is somewhat misleading.]
C. J. Bouwkamp (Eindhoven).

Bouwkamp, C. J. On Sommerfeld's surface wave. Physical Rev. (2) **80**, 294 (1950).

The author elaborates his criticisms of a series of contributions of Kahan and Eckart [C. R. Acad. Sci. Paris **227**, 969–970 (1948); J. Phys. Radium (8) **10**, 165–176 (1949); same Rev. (2) **76**, 406–410 (1949); these Rev. **11**, 143] to the theory of the radiating dipole over an infinite plane earth. These were (a) the remark that Sommerfeld's surface wave does not satisfy the radiation condition, (b) a correction to Sommerfeld's approximate evaluation of his solution, and (c) a proof that Sommerfeld's radiation condition ensures uniqueness for real k_1 and k_2. It is pointed out that (a) is irrelevant, that (b) is not new, and that (c) is fallacious. It is suggested that the radiation condition in Sommerfeld's form does not apply to an infinite plane earth.
F. V. Atkinson (Ibadan).

Eckart, G., et Kahan, T. Sur la réflexion "interne" dans un milieu stratifié. Application particulière à la troposphère. J. Phys. Radium (8) **11**, 569–576 (1950).

Brehovskih, L. M. The reflection of plane waves from stratified nonhomogeneous media. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **19**, 1126–1135 (1949). (Russian)

The author derives the Riccati-type equation

$$dV/dz = 2\mathfrak{J}(V) + \gamma(1-V^2)$$

for the reflection coefficient $V(z)$ for the passage of monochromatic waves through a plane-stratified medium. The waves may be electromagnetic with normal or with tangential electric vector, the equation in these cases being attributed to van Cittert [Physica **6**, 840–848 (1939)], or may be acoustic. Two methods are proposed for using Picard's method to solve the equation approximately, with a view to numerical applications. The second of these methods takes as first approximation the solution of the differential equation ignoring the term in V^2, and appears to give good results in a particular numerical case. The procedure is thought to be applicable to cases in which the parameters of the medium fluctuate. *F. V. Atkinson.*

Tihonov, A. N., and Muhina, G. V. Determination of a variable electric field in a stratified medium. Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **14**, 99–112 (1950). (Russian)

Maxwell's equations are solved, with the aid of Bessel transforms of order zero and one, in the case of a dipole on the x-axis located at the origin in the plane $z=0$, face of an infinite conducting layer of thickness l of conductability $\sigma\neq 0$ bounded by two parallel planes $z=0$, $z=-l$ with $\sigma=0$ for $z>0$ and $z<-l$. The expressions and the graphs of amplitude and phase as functions of x $(y=0, z=0)$ are given for various values of l for the x-component E_x of the electrical field created by the dipole. *E. Kogbetliantz.*

Tihonov, A. N., and Skugarevskaya, O. A. On the establishment of an electric current in a nonhomogeneous stratified medium. II. Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **14**, 281–293 (1950). (Russian)

[For part I see the same vol., 199–222 (1950); these Rev. **12**, 65.] Notwithstanding the title of this paper, the solution of Maxwell's equations is given only for a homogeneous unstratified layer (constant σ) of finite thickness. The component of the electrical field in the direction of the dipol which generates this field is computed with the aid of an approximate formula which can yield this component with any prescribed accuracy. The integrals Q_1 and Q_2 (p. 291) into the sum of which is decomposed the functions $Q=2(Q_1+Q_2)/\sigma$ have no meaning since $Q_1=+\infty$, $Q_2=-\infty$, both being essentially divergent (nonsummable) integrals. Nevertheless, the results are exact. *E. Kogbetliantz.*

Kogan, S. H. The excitation of a spiral line. Doklady Akad. Nauk SSSR (N.S.) **74**, 489–492 (1950). (Russian)

The propagation of electromagnetic waves along a spiral [see the same Doklady (N.S.) **66**, 867–870 (1949); these Rev. **10**, 764] is further investigated by considering the influence function under external excitation of a differential element of the spiral wire, and generalizing the boundary conditions by taking finite electrical conductivity into account. The Fourier integral solution thus modified can be written as a contour integral which is reduced to pole terms and branch integrals. The latter represent attenuated wave components and can be neglected. For the former, representing the practically important unattenuated wave type, approximative expressions are obtained in characteristic frequency regions, which are in satisfactory agreement with experiment. *H. G. Baerwald* (Cleveland, Ohio).

Netušil, A. V. Electric fields in anisotropic media. Élektričestvo **1950**, no. 3, 9–19 (1950). (Russian)

Two cases of electric fields in nonisotropic media are considered. The dielectric properties of the medium are described by a tensor of the second rank diagonal in: (a) Cartesian coordinates; (b) cylindrical coordinates. Both problems are reduced to the isotropic case by a suitable transformation. Several special cases are considered: two charged wires; a semiconducting plate; capacitance of a two wire line near an anisotropic dielectric; parallel plate condenser with a nonisotropic dielectric; anisotropic cylinder in a uniform field; a system of cylindrical electrodes surrounded by coaxial layers of isotropic and nonisotropic dielectrics; losses in an anisotropic dielectric in a high frequency field. *G. M. Volkoff* (Vancouver, B. C.).

Logunov, A. A. On spatial periodicity of a gaseous discharge. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **20**, 458–464 (1950). (Russian)

An equation for the distribution function of electrons in the plasma is set up. It takes into account: electron recombination; ionization and excitation of the molecules and atoms of the gas; and scattering collisions of electrons with atoms and molecules. The important role of distant interactions in giving rise to periodic solutions is brought out. A special case is considered in detail which takes only electron diffusion into account in investigating the spatial periodicity of the gaseous discharge. In the linear approximation periodic solutions are obtained for the electron density. *G. M. Volkoff* (Vancouver, B. C.).

v. Laue, M. Zur Minkowskischen Elektrodynamik der bewegten Körper. Z. Physik **128**, 387–394 (1950).

Minkowski's electrodynamics of a moving medium with dielectric constant ϵ and magnetic permeability μ uses a nonsymmetric stress energy tensor τ_{ij} and therefore distinguishes between the momentum of the electromagnetic field and the flow of energy density. The author shows that for this tensor, in contrast to that proposed by Abraham, when the field is described by plane waves the three-dimensional vector whose components are $\tau_{4\alpha}/\tau_{44}=w_\alpha$ behaves as a velocity in the sense that under a Lorentz transformation it satisfies the addition theorem for velocities.
A. H. Taub (Urbana, Ill.).

Géhéniau, Jules. Solutions singulières de l'équation de Klein-Gordon tenant compte d'un champ magnétique extérieur. C. R. Acad. Sci. Paris **231**, 610–612 (1950).

The author constructs the Hadamard elementary solution for the Klein-Gordon equation for a particle in a constant uniform magnetic field. *A. H. Taub* (Urbana, Ill.).

Slepian, Joseph. Electromagnetic ponderomotive forces within material bodies. Proc. Nat. Acad. Sci. U. S. A. **36**, 485–497 (1950).

The postulate of surface and volume forces within material bodies in an electromagnetic field is critically analysed and it is shown that no physically significant and uniquely definable set of such forces exists. Choosing as a possible electromagnetic stress tensor any one which in empty space reduces to Maxwell's classical stress tensor, it is proposed to define the associated mechanical stress tensor through the vector difference between the calculated electrical surface or volume forces and the actually observed mechanical forces, this being derivable from a tensor. In this manner, an expression for the net volume force is derived which includes all the previously used terms and additional ones.
E. Weber (Brooklyn, N. Y.).

Baudoux, Pierre. Note sur les systèmes modernes d'unités électriques. Acad. Roy. Belgique. Cl. Sci. Publ. Fond. Agathon De Potter no. 3, 15 pp. (1950).

This note discusses the relations between electromagnetic systems by introducing an additional arbitrary constant into Maxwell's field equations. *E. Weber.*

Standards on circuits: definitions of terms in network topology, 1950. Proc. I.R.E. **39**, 27–29 (1951).

Zimmermann, F. Die Auflösung knotenpunktsbelasteter elektrischer Netze mittels Matrizen. Österreich. Ing.-Arch. **4**, 243–251 (1950).

Matrix algebra is applied to the problem of determining the voltages in an electric network without mutual coupling when certain currents are prescribed. Let the network consist of n branches and p vertices, and suppose that the "driving currents" applied at each vertex are known. Then the voltage differences u' between a designated vertex, say the pth, and the remaining $(p-1)$ vertices may be found from the following considerations. The voltage differences, u, across each branch may be expressed in terms of u' through a transformation matrix $\mathbf{c}$ whose elements are selected from 0, -1, or 1 in accordance with the network topology and assigned positive directions. If $\mathbf{U}$ represents the column matrix of u, $\mathbf{U}'$ the column matrix of u', then $\mathbf{U}=\mathbf{c}\mathbf{U}'$. If $\mathbf{I}$ represents the column matrix of the branch currents I, and $\mathbf{I}'$ the corresponding matrix of the "driving currents" I', then power loss invariance requires $\mathbf{I}'=\bar{\mathbf{c}}_t\mathbf{I}$, where $\bar{\mathbf{c}}_t$ is the transpose of $\mathbf{c}$ with conjugate elements. The branch currents $\mathbf{I}$ and branch voltage differences $\mathbf{U}$ are related through the admittances $\mathbf{Y}$, as $\mathbf{I}=\mathbf{Y}\mathbf{U}$, whence it follows from the foregoing expressions that $\mathbf{I}'=\bar{\mathbf{c}}_t\mathbf{Y}\mathbf{c}\mathbf{U}'=\mathbf{Y}'\mathbf{U}'$. Thus the solution is $\mathbf{U}'=(\mathbf{Y}')^{-1}\mathbf{I}'$. Important special cases occur when one or more of the $(p-1)$ vertices is connected to the pth vertex, or when one or more branches may be missing. These are easily handled. *R. Kahal.*

Kafka, Heinrich. Neue Leitwertdiagramme für passive lineare Vierpole. Arch. Elektr. Übertragung **4**, 446–454 (1950).

The solution of certain four-terminal electrical network problems is presented in graphical form. In engineering terminology, vector diagrams for certain desired quantities such as transfer admittance, power transfer ratio, voltage transfer ratio, driving point admittance, etc., are given for three different conditions. In the three cases treated the assumed known quantities are respectively: (1) the output voltage and terminating admittance; (2) the input voltage and driving point admittance; and (3) the input voltage and terminating admittance. *R. Kahal* (St. Louis, Mo.).

Quantum Mechanics

Cooper, J. L. B. The paradox of separated systems in quantum theory. Proc. Cambridge Philos. Soc. **46**, 620–625 (1950).

This paper deals with what the author calls the "paradox" discussed by Einstein, Podolsky, and the reviewer [Physical Rev. (2) **47**, 777–780 (1935)] in which two quantum-

mechanical systems O and I, after interacting, are left in such a state that, by measuring the coordinate or the momentum of I, one can determine the coordinate or the momentum of O without disturbing O, in spite of the fact that the coordinate and the momentum do not commute in the quantum theory. The aim of this work is to show that the arguments on which the paradox is based are not valid within the framework of the nonrelativistic theory. According to the author, if there is no barrier between O and I, there is a finite probability that I will move to O and disturb it (by interference) within any time interval, no matter how short, after a measurement on I is made. On the other hand, if an impenetrable barrier is placed between O and I, the operators representing the momenta are no longer self-adjoint, and hence the expansion involving their eigenfunctions which was used in the argument of the paradox is not valid. *N. Rosen* (Chapel Hill, N. C.).

Gamba, A. The uncertainty relation. Nature **166**, 653–654 (1950).

Dacev, Asen. The principle of indeterminacy in contemporary physics. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. **45**, 203–226 (1949). (Bulgarian)
Expository lecture.

Soh, Hsin P. Examples on the calculation of energy states by the uncertainty relations. Philos. Mag. (7) **41**, 851–854 (1950).

Using the principle that the minimum values of the coordinates and momenta of a quantum-mechanical system may not be less than the uncertainties of these quantities as given by the Heisenberg uncertainty principle, the author derives in a simple way lower bounds for the minimum energies of atomic systems such as the H, He, Li, Be, B, and C atoms. The agreement with experimental minima is quite close, and may be improved further by refinements of the method. *O. Frink* (State College, Pa.).

Nanda, V. S. A note on Weyl's inequality. Indian J. Phys. **24**, 181–184 (1950).

The author gives an elementary proof of the following: If p and q are operators satisfying $qp-pq=i\hbar$, then

$$e^{\alpha p+\beta q}=e^{\alpha p}e^{\beta q}e^{\frac{1}{2}i\hbar\alpha\beta}=e^{\beta q}e^{\alpha p}e^{-\frac{1}{2}i\hbar\alpha\beta},$$

where α and β are ordinary numbers. *A. H. Taub.*

Krupp, Helmar. Bestimmung der allgemeinen Lösung der Schrödinger-Gleichung für Coulomb-Potential. Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. **97**, no. 8, 28 pp. (1950).

The solutions of the radial wave equation of the wave-mechanical Kepler problem that satisfy the usual boundary conditions (R finite at $r=0$, $R\rightarrow 0$ for $r\rightarrow\infty$) are well known. In various problems of physical interest there is need for solutions that do not satisfy the boundary conditions; this is, of course, always the case for one of the two solutions, and it is so for both except when the energy has one of its characteristic values. The author chooses explicit definitions for the two solutions, and tabulates them numerically in the cases $l=0$: $n=\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}$; $l=1$: $n=\frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}$; $l=2$: $n=\frac{5}{2}, 3, \frac{7}{2}$. *W. H. Furry* (Cambridge, Mass.).

Mayot, Marcel. Le calcul des perturbations en mécanique quantique. C. R. Acad. Sci. Paris **231**, 1426–1428 (1950).

Pluvinage, Philippe. Sur une singularité des fonctions d'onde des atomes à deux électrons. C. R. Acad. Sci. Paris **231**, 823–825 (1950).

✱Gombás, P. Theorie und Lösungsmethoden des Mehrteilchenproblems der Wellenmechanik. Verlag Birkhäuser, Basel, 1950. 268 pp.

This book aims at giving as simple as possible a presentation of the many particle problem in wave mechanics (part one), and of the various approximation procedures used in solving it (part two). In part one, chapters I–II, the general theoretical foundations are outlined, and in chapters III–IV they are applied to the theory of simple atoms and molecules. Chapters V and VI deal with quantum statistics and second quantisation. In part two various special methods of obtaining approximate solutions of the many particle problem are discussed in some detail. Chapter VII, which comprises about 30% of the text, deals with the variational procedure and its most important practical applications in the theory of atoms and molecules. In the last two paragraphs of this chapter a perturbation procedure is developed which is based upon the variational method, and it is used to calculate the polarizability of atoms. Chapter VIII describes Hartree's method of the self-consistent field and its generalisation by Fock. Finally, in chapter IX, the statistical method of Thomas and Fermi, and its modifications by Lenz, Jensen, and the author, is presented. This book is a good introduction, and it might also prove of considerable help to the specialist who is interested in practical applications of the many particle theory. Part two brings not only the standard methods but also a survey of not so well-known modifications which heretofore could be found only in the original papers. *E. Gora* (Providence, R. I.).

Sáenz, A. W. On time-independent integrals of motion of the one-body problem in Dirac theory. Physical Rev. (2) **79**, 1004–1005 (1950).

Riss, Marsel' [Riesz, Marcel]. On some fundamental notions of relativistic quantum mechanics. Uspehi Matem. Nauk (N.S.) **5**, no. 5(39), 120–144 (1950). (Russian)

Translated from C. R. Dixième Congrès Math. Scandinaves 1946, pp. 123–148; these Rev. **8**, 427.

Costa de Beauregard, Olivier. Sur la symétrie relativiste dans le formalisme non superquantifié. C. R. Acad. Sci. Paris **231**, 1423–1425 (1950).

Nagakura, T. Some remarks on the relativistic quantum field theory. Progress Theoret. Physics **5**, 502–503 (1950).

The Schrödinger equation of a general quantized field theory is derived in a formally covariant way. The resulting equation has the form (1) $(H(P,\sigma)-i\hbar(\partial/\partial\sigma_P))\Psi(\sigma)=0$, $H(P,\sigma)=-n^\mu n_\nu T_\mu{}^\nu$, where $T_\mu{}^\nu$ is the ordinary energy-momentum tensor of the fields, and n^μ, n_ν are components of the unit vector normal to the space-like surface σ at the point P. General commutation laws for the field variables are also derived. In this short letter the limitations of equation (1) are not made clear. This equation has been independently derived, and its meaning much more fully discussed, by de Wet [Proc. Roy. Soc. London. Ser. A. **201**, 284–296 (1950); these Rev. **11**, 633]. *F. J. Dyson.*

Utiyama, Ryoyu. On the covariant formalism of the quantum theory of fields. I. Progress Theoret. Physics 5, 437–458 (1950).

This is the first part of a paper presenting a complete and ab initio development of the covariant quantum field theory, attention being confined to the purely formal aspects of the theory. There are five sections, with subject matter as follows. (1) Expression of a classical field theory, based on a given Lagrangian, in terms of general curvilinear coordinate systems; the resulting theory is invariant under two separate groups of transformations, the Lorentz transformations of the space, and all transformations of the coordinate system. (2) The conservation laws which are deducible in the classical theory from the invariance of the formalism under various types of infinitesimal transformations. (3) The quantization of the field is achieved by imposing canonical commutation relations in one coordinate system, and then proving these relations to be invariant under coordinate transformations; the equations of motion appear in canonical form, with a Hamiltonian operator which depends on the coordinate system. (4) The equations of motions are written in a covariant Hamiltonian form by passing to the super-many-time theory of Tomonaga [same journal 1, 27–42 (1946); these Rev. 10, 226]. (5) The interaction representation is defined, and the Schrödinger equation in this representation (the Tomonaga-Schwinger equation) is derived in such a way that the integrability of the equation is automatically assured and does not need to be verified. *F. J. Dyson* (Birmingham).

Sato, Iwao. General-relativistic quantum theory of wave field. II. Sci. Rep. Tôhoku Univ., Ser. 1. 33, 136–143 (1950).

The major part of this paper is concerned with the working-out of a properly covariant formalism to describe the motion of a charged scalar field interacting with the electromagnetic field in a general space-time with a Riemannian metric. The two fields are both quantized, but the metric tensor is treated classically. So long as the metric is supposed to be given a priori, this formalism raises no particular difficulties of interpretation. At the end of the paper, following a program announced previously [same vol., 30–37 (1949); these Rev. 12, 149], it is proposed to take into account the reaction of the quantized fields upon the metric by making the metric satisfy the classical Einstein field equations with the matter-tensor given by the expectation-value of the energy-momentum tensor of the quantized fields. The whole formalism then becomes nonlinear in the state-vector of the system, which makes its physical interpretation very problematical.

F. J. Dyson (Birmingham).

Ôneda, Sadao, and Ozaki, Shoji. On the relativistic covariance of the self-energy of an electron. Sci. Rep. Tôhoku Univ., Ser. 1. 33, 25–29 (1949).

The mass correction $W(p)$ of a moving electron on account of its electromagnetic field is known not to be related to $W(0)$ by $W(p)=W(0)(1-\beta^2)^{\frac{1}{2}}$, $\beta=v/c$. This is due to the effect of motion on the angular distribution of photons emitted virtually by the electron. If it is assumed that the observed mass of the electron is the sum of the mechanical mass μ and a contribution $\Delta\mu$ from the meson field, a relation $W(p)/W(0)=\mu/(p^2+\mu^2)^{\frac{1}{2}}$ is obtained. However, the correction derived by perturbation methods from the c-meson field theory of Sakata [Progress Theoret. Physics 2, 30–31 (1947)] and Pais [Physical Rev. (2) 68, 227–228 (1945)] and hole theory does not transform in this way. A further calculation is given using Tomonaga's super-many-time theory and canonical transformation. For $p=0$ this reproduces the $W(0)$ given as above. There is a discrepancy between this and $\lim_{p\to 0} W(p)$, where $W(p)$ is obtained by Tomonaga's prescription. Thus the relativistic covariance of self-energy needs more profound consideration.

C. Strachan (Aberdeen).

Sawada, K. Structure of electron in λ-process. Progress Theoret. Physics 5, 497–498 (1950).

The author analyzes the Dirac theory of the classical electron [Proc. Roy. Soc. London. Ser. A. 180, 1–40 (1942); these Rev. 5, 277] usually known as the λ-limiting theory. He shows that the theory, before passing to the limit $\lambda=0$, is equivalent to the assumption of a definite finite size and shape for the charge-distribution of the electron. This concrete model of the electron makes it clear why the self-energy vanishes identically. *F. J. Dyson* (Birmingham).

Nishijima, Kazuhiko. On the elimination of the normal-dependent part from the Hamiltonian. Progress Theoret. Physics 5, 405–411 (1950).

A new proof, of more general applicability than previously published discussions, is given for the theorem that the S-matrix calculated in field theory as a power series in the interaction depends directly on the interaction Lagrangian and not on the interaction Hamiltonian. This result simplifies considerably the practical calculation of the S-matrix. The explicit expression for the S-matrix in terms of the interaction Lagrangian L is obtained in the form

$$S=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{\hbar}\right)^n\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}P^*(L(x_1),\cdots,L(x_n))\,dx_1\cdots dx_n,$$

where P^* represents a modification of the chronological ordering operation used by the reviewer [Physical Rev. (2) 75, 486–502 (1949); these Rev. 10, 418]. *F. J. Dyson*.

Nakabayasi, Kugao, and Sato, Iwao. On the elimination of the surface-dependent electromagnetic interactions in meson theory. Sci. Rep. Tôhoku Univ., Ser. 1. 34, 1–4 (1950).

It is proved that in the calculation of the S-matrix by perturbation theory, for a system of three fields in interaction, the surface-dependent terms in the interaction Hamiltonian make no effective contribution and can in practice be ignored. The example discussed in detail is a charged pseudoscalar meson field interacting with a charged nucleon field and with the electromagnetic field. The proof is a direct extension of that given by Matthews [Physical Rev. (2) 76, 684–685 (1949)] for the case of two fields.

F. J. Dyson (Birmingham).

Kanesawa, S. On mixed field theories and vacuum polarization. Progress Theoret. Physics 5, 492–494 (1950).

When the polarization of the vacuum in quantum electrodynamics is calculated by second-order perturbation theory, two types of divergence are found, the first representing a photon self-energy and the second a charge-renormalization effect. Using a charged scalar field in order to compensate these divergences, Feldman [Physical Rev. (2) 76, 1369–1375 (1949); these Rev. 12, 150] found that the first could be made to vanish but the second could not. The introduction of charged vector fields was also insufficient to compensate

all the infinities. The author here extends Feldman's work to include a charged field of spin 3/2, using the formalism of Harish-Chandra [Proc. Roy. Soc. London. Ser. A. **192**, 195–218 (1948); these Rev. **9**, 557]. The result is that the divergences due to the spin $-3/2$ field are of the same type as those due to a vector field with vector and tensor coupling; but a complete compensation is again impossible because the signs of some of the terms are such as to reinforce and not to cancel each other. *F. J. Dyson.*

Watson, Kenneth M., and Hart, Edward W. On the use of the Tomonaga intermediate coupling method in meson theory. Physical Rev. (2) **79**, 918–925 (1950).

The Tomonaga intermediate coupling method [S. Tomonaga, Progress Theoret. Physics **2**, 6–24 (1947)] is presented in a simplified form. Application is made to the problems of nuclear forces and photomeson production. The latter is particularly interesting since it is shown that this is one of the few cases where exact information as to the wave function of a free meson is available. The results obtained agree with known strong and weak coupling limits and seem to join the two smoothly. *K. M. Case.*

Hönl, Helmut, und Boerner, Hermann. Zur de Broglie'schen Theorie der Elementarteilchen. Z. Naturforschung **5a**, 353–366 (1950).

The method of de Broglie [Théorie générale des particules à spin, Méthode de fusion, Gauthiers-Villars, Paris, 1943] for the construction of relativistic wave equations is applied to the case of fusion of two and three particles. The resulting matrices are obtained in reduced form by direct construction. It is assumed that the irreducible components may be taken to describe elementary particles. For the fusion of two particles the results of Kemmer [Proc. Roy. Soc. London. Ser. A. **173**, 91–116 (1939); these Rev. **1**, 95] are refound. For the triple fusion the irreducible equations describe: (a) a Dirac type particle of spin $\frac{1}{2}$; (b) a particle described by a 16 component wave function which can exist in states of spin $\frac{1}{2}$ and $\frac{3}{2}$ and masses in the ratio 1:3; (c) a particle described by a 20 component wave function with properties similar to (b). The magnetic moments for particles described by these equations are obtained. No particular connection with experimental nucleon moments is obtained. *K. M. Case* (Ann Arbor, Mich.).

Kyu, Gakkei, and Ozaki, Shoji. On the mixed field theory in the decay of a π-meson into a μ-meson and neutrino. Sci. Rep. Tôhoku Univ., Ser. 1. **33**, 133–135 (1950).

The matrix elements for the decay of a π-meson into a μ-meson and a neutrino are calculated, assuming that the decay takes place not by a direct coupling but through the intermediary of a nucleon field which is coupled both to the π-meson and to the μ-meson-neutrino fields. The matrix elements are divergent, however the character of the π-meson and of the interaction is choosen. In order to obtain finite matrix elements, the authors suppose that the mesons are coupled simultaneously to two separate nucleon fields, one having the usual Dirac character and the other being a scalar field; this idea of compensating divergences by introducing an additional scalar field is due to Pais and Sakata [Progress Theoret. Physics **2**, 145–150 (1947)]. It is found that the compensation succeeds when the π-meson is scalar, but fails for a pseudo-scalar or vector π-meson.

F. J. Dyson (Birmingham).

Wick, G. C. The evaluation of the collision matrix. Physical Rev. (2) **80**, 268–272 (1950).

Following an idea of Houriet and Kind [Helvetica Phys. Acta **22**, 319–330 (1949); these Rev. **11**, 301] the author rearranges the factors occurring in Dyson's expression for the S-matrix so that it becomes a sum of terms in each of which all creation operators are to the left of all annihilation operators. This has the advantage of automatically "digesting" all creation-followed-by-annihilation processes. Two theorems and a number of rules by which the rearrangement may be effected are proved. Feynman's rules [same Rev. (2) **76**, 749–759 (1949)] for evaluating the collision matrix follow immediately. *A. J. Coleman* (Toronto, Ont.).

Kinoshita, Toichiro. On the interaction of mesons with the electromagnetic field. I. Progress Theoret. Physics **5**, 473–488 (1950).

The equations describing the interaction of mesons of spin 0 and 1 with the electromagnetic field are set up in the interaction representation. Use is made of the Kemmer-Duffin formalism. The formal S matrix is set up to permit the calculation of radiative corrections. In this paper only the second order correction to the mesonic charge-current caused by the vacuum polarization terms are found. Other applications are promised for a later paper.

K. M. Case (Ann Arbor, Mich.).

Gupta, Suraj N. On the interaction of vector mesons with nucleons. Proc. Cambridge Philos. Soc. **46**, 649–650 (1950).

A new notation is introduced which simplifies the form of the interaction between vector mesons and nucleons, when the meson field is described by a Duffin-Kemmer matrix formalism [Kemmer, Proc. Roy. Soc. London. Ser. A. **173**, 91–116 (1939); these Rev. **1**, 95]. *F. J. Dyson.*

Case, K. M., and Pais, A. On spin-orbit interactions and nucleon-nucleon scattering. Physical Rev. (2) **80**, 203–211 (1950).

The authors propose a very short range nuclear interaction of the (**L**·**S**) form as providing a charge independent description of nucleon-nucleon scattering. Pauli principle restrictions on the proton-proton system cause the proposed interaction to play a much larger role for it than for the neutron-proton system, permitting an explanation of the observed differences between the high energy scattering of protons by protons and by neutrons. To fit the proton-proton scattering at both 30 mev and 350 mev it is found necessary to assume a shape for the potential function which is strongly singular at $r=0$, and which has a small tail. Calculations are performed entirely in Born approximation, so that the quantitative accuracy of the paper is uncertain.

N. Austern (Madison, Wis.).

Sato, Iwao. On the anomalous magnetic moment of the nucleon. Sci. Rep. Tôhoku Univ., Ser. 1. **33**, 83–91 (1949).

Magnetic moments of proton and neutron are calculated by second-order perturbation theory, assuming the meson field to be pseudo-scalar, charged, or symmetrical, with pseudo-vector coupling. The results are finite but in disagreement with experiment. They are in such a form that a direct comparison with the independent calculations of K. M. Case [Physical Rev. (2) **76**, 1–13 (1949)] is not possible. *F. J. Dyson* (Birmingham).

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